Homological properties of modules over semigroup algebras

September 13, 2009

Paul Ramsden

Abstract

Let S be a semigroup. In this paper we investigate the injectivity of $\ell^1(S)$ as a Banach right module over $\ell^1(S)$. For weakly cancellative S this is the same as studying the flatness of the predual left module $c_0(S)$. For such semigroups S, we also investigate the projectivity of $c_0(S)$. We prove that for many semigroups S for which the Banach algebra $\ell^1(S)$ is non-amenable, the $\ell^1(S)$-module $\ell^1(S)$ is not injective. The main result about the projectivity of $c_0(S)$ states that for a weakly cancellative inverse semigroup S, $c_0(S)$ is projective if and only if S is finite.

1 Introduction

Let S be a semigroup, and let $\ell^1(S)$ be its associated Banach convolution algebra. In this paper we study certain homological properties of modules over $\ell^1(S)$. The aim is to characterize homological properties of the Banach algebra $\ell^1(S)$ (and its modules) in terms of the underlying semigroup S. Homological properties of Banach algebras associated with groups and semigroups have been studied by many authors. some recent papers are, [1, 6, 7, 8].

The notions of *projectivity*, *injectivity*, and *flatness* are fundamental in homology theory. The theory of these concepts in the category of Banach modules is expounded by A. Ya. Helemskii in [10]. In [4], H. G. Dales and M. E. Polyakov undertook a study of these properties for various canonical modules associated with the group algebra $L^1(G)$. For example, they proved the following theorem. Here we regard $C_0(G)$ as a submodule of the dual module $L^\infty(G) = L^1(G)'$.

Theorem 1.1 ([4, Theorems 3.1, 4.7 and 4.9]). *Let G be a locally compact group. Then:*
(i) *$L^1(G)$ is an injective right $L^1(G)$-module if and only if G is discrete and amenable.*
(ii) *$C_0(G)$ is a flat left $L^1(G)$-module if and only if G is amenable.*
(iii) *$C_0(G)$ is a projective left $L^1(G)$-module if and only if G is compact.* □

In this paper we shall investigate the same questions for the Banach algebra $\ell^1(S)$, where S is a (discrete) semigroup. We cannot hope for such simple characterizations for a general semigroup. This is partly due to the complexities of describing the amenability of the Banach algebra $\ell^1(S)$ in terms of the semigroup S; see [3, Theorem 10.12].

Overview of contents

- Section 2 contains all the background material and definitions about Banach modules and semigroups that we shall need.
- In Section 3 we prove some general results about injective Banach modules.
- In Section 4 we show how amenability of the underlying semigroup enters the picture. Here we show that the injectivity of certain modules over $\ell^1(S)$ implies that S

is amenable. This follows from a general result which also applies to modules over $L^1(G)$ and $M(G)$ for a locally compact group G. In Theorem 4.10, under the additional assumption that S is cancellative, we give a characterization the injectivity of $\ell^1(S)$ (S must be an amenable group). At the end of this section we give an example (Example 4.12) of a finite semigroup S such that $\ell^1(S)$ is a right injective module, but $\ell^1(S)$ is not amenable.

- In Section 5 we investigate the flatness of the left $\ell^1(S)$-predual module $c_0(S)$ for a weakly right cancellative semigroup S. Our main theorem (Theorem 5.5) gives a necessary combinatorial condition on the set of idempotents. This condition is not satisfied by the bicyclic semigroup or $(\mathbb{N}, \max)$.
- In Section 6 we move on to investigating the semigroups S such that $c_0(S)$ has the stronger property of being projective. Here we prove, in Theorem 6.5 that, if S is a weakly cancellative semigroup such that $c_0(S)$ is projective, then S must be finite. An immediate corollary of this result (Theorem 6.6) gives a characterization for the class of inverse semigroups ($c_0(S)$ is projective if and only if S is finite).

2 Preliminaries

For $n \in \mathbb{N}$, we set $\mathbb{N}_n = \{1, 2, \ldots, n\}$. The *indicator function* of a subset T of a set S is denoted by χ_T. Let $f : S \to E$ be a function from a set S to a vector space E. For $T \subset S$ we define $\chi_T f : S \to E$ by $(\chi_T f)(s) = f(s) \ (s \in T)$ and $(\chi_T f)(s) = 0 \ (s \in S \setminus T)$.

Let E be a Banach space. We denote the dual space by E'; the action of $\lambda \in E'$ on an element $x \in E$ is written as $\langle x, \lambda \rangle$. For a subspace F of a Banach space E we set

$$F^0 = \{\lambda \in E' : \langle x, \lambda \rangle = 0 \ (x \in F)\} .$$

Then F^0 is a closed subspace of E'. We set $F^{00} = (F^0)^0 \subset E''$.

The projective tensor product is denoted by $\widehat{\otimes}$, and $\mathcal{B}(E, F)$ denotes the space of all bounded linear operators between Banach spaces E and F.

2.1 Banach modules

Throughout this section A is a Banach algebra. We denote by A-**mod**, and by **mod**-A the categories of Banach left A-modules, and of Banach right A-modules, respectively. We shall give definitions only for one category of module; similar definitions apply for modules in other categories.

Let $E \in A$-**mod**. Then the dual space E' has a natural right A-module structure given by

$$\langle x, \lambda \cdot a \rangle = \langle a \cdot x, \lambda \rangle \quad (a \in A, x \in E, \lambda \in E') .$$

This module is called is the *dual module* of E.

Let $S \subset A$ be a subset, and let $E \in A$-**mod**. We set

$$S \cdot E = \{a \cdot x : a \in S, x \in E\} \quad \text{and} \quad SE = \text{lin}\, S \cdot E ,$$

the linear span of $S \cdot E$. If S is a left ideal of A then SE is a submodule of E. The *essential part* of E is the closed submodule $\overline{AE}$, and E is *essential* if $\overline{AE} = E$. Now we describe the 'dual' concept. Let $F \in$ **mod**-A. For a subset $S \subset A$ we set

$$F^{\perp S} = \{x \in F : x \cdot S = \{0\}\} .$$

If S is a left ideal of A then $F^{\perp S}$ is a closed submodule of F. We set $F^{\perp} = F^{\perp A}$, which is the *annihilator submodule* of F. The A-module F is *faithful* if $F^{\perp} = \{0\}$, and F is an *annihilator module* if $F^{\perp} = F$. Now suppose that $F = E'$ for some $E \in A$-**mod**. Then $F^{\perp S} = (\overline{SE})^{\,0}$, and so we have

$$(\overline{SE})' = F/(\overline{SE})^{\,0} = F/F^{\perp S}.$$

Hence, if $\overline{SE} = E$, then $F^{\perp S} = \{0\}$. In particular the dual of an essential module is faithful. Similarly, for $E \in A$-**mod** we set

$${}^{S\perp}E = \{x \in E : S \cdot x = \{0\}\}\,.$$

For $E, F \in A$-**mod** we denote by ${}_A\mathcal{B}(E, F)$ the subspace of $\mathcal{B}(E, F)$ consisting of bounded left A-module morphisms. Similarly, $\mathcal{B}_A(E, F)$ denotes the space of bounded right A-module morphisms when $E, F \in$ **mod**-A.

The unitization of A is denoted by $A^\flat$. If $E \in A$-**mod**, then we consider $E \in A^\flat$-**mod** in the obvious way.

Let $E \in A$-**mod**. Then $A^\flat \widehat{\otimes} E \in A$-**mod** with module operation specified by

$$a \,\cdot\, (b \otimes x) = ab \otimes x \quad (a \in A,\, b \in A^\flat,\, x \in E)\,.$$

We define the *canonical morphism* $\pi : A^\flat \widehat{\otimes} E \to E$ by

$$\pi(a \otimes x) = a \,\cdot\, x \quad (a \in A^\flat,\, x \in E)\,.$$

Let $E \in$ **mod**-A. Then $\mathcal{B}(A^\flat, E) \in$ **mod**-A with the module operation

$$(T \cdot a)(b) = T(ab) \quad (a \in A,\, b \in A^\flat,\, T \in \mathcal{B}(A^\flat, E))\,.$$

We define the *canonical embedding* $\Pi : E \to \mathcal{B}(A^\flat, E)$ by the formula

$$\Pi(x)(a) = x \cdot a \quad (a \in A^\flat,\, x \in E)\,.$$

2.2 Banach homology

We now recall the definitions and basic relationships from Banach homology that this paper is concerned with. For full details see [10] and [11]. We eschew the original homological definitions and give the standard characterizations which have proved most useful for checking projectivity and injectivity in specific cases. Because of the duality involved it is convenient to study injective right modules and projective and flat left modules.

Proposition 2.1. *Let A be a Banach algebra.*

(i) *Let $F \in$ **mod**-A. Then F is* injective *if and only if there exists $\rho \in \mathcal{B}_A(\mathcal{B}(A^\flat, F), F)$ with $\rho \circ \Pi = I_F$.*

(ii) *Let $E \in A$-**mod**. Then E is* projective *if and only if there exists $\rho \in {}_A\mathcal{B}(E, A^\flat \widehat{\otimes} E)$ with $\pi \circ \rho = I_E$.*

(iii) *Let $E \in A$-**mod**. Then E is* flat *if and only if there exists $\rho \in {}_A\mathcal{B}(E, (A^\flat \widehat{\otimes} E)'')$ such that the following diagram commutes*

$$\begin{array}{ccc} E & \xrightarrow{\ \rho\ } & (A^\flat \widehat{\otimes} E)'' \\ & \searrow^{\iota_E} & \downarrow{\scriptstyle \pi''} \\ & & E'' \end{array} \quad .$$

(iv) *If either E is essential or F is faithful, then we can replace $A^\flat$ by A in the above characterizations.* □

Part (i) in proved in [10, III.1.31] and the case where E is faithful is [4, Proposition 1.7]; part (ii) is [10, IV.1.1, IV.1.2]; part (iii) is similar to [10, Exercise VII2.8], see also [15, Lemma 4.3.22].

The following facts are elementary: a module E is flat if and only if the dual module E' injective; every projective module is flat.

The classes of *amenable* and *contractible* Banach algebras are particularly important and well studied (see [10, Chapter IV], [12] or [2, §2.8]). The following proposition gives one nice property of these Banach algebras.

Proposition 2.2. *Let A be a Banach algebra, and let $E \in A$-**mod** or $E \in$ **mod**-A.*

(i) *If A is contractible, then E is projective.*

(ii) *If A is amenable, then E is flat.* □

The following is a long standing open problem.

Question 2.3. *Let A be a Banach algebra such that every $E \in A$-**mod** and every $E \in$ **mod**-A is flat [projective]. Is A amenable [contractible]?*

The answer is known to be positive for many classes of Banach algebras, in particular for the class of group algebras. Part of the motivation for this work was to answer this question in the class of semigroup algebras. Although we come short of a complete answer, our results strongly suggest that the answer is 'yes' in this class.

2.3 Semigroups and semigroup algebras

Let S be a semigroup. The set of idempotents in S is denoted by $E(S)$. A semigroup S is a *semilattice* if S is commutative and $E(S) = S$. The *canonical partial order* on $E(S)$ is given by

$$p \leq q \Longleftrightarrow p = pq = qp \quad (p, q \in E(S)) .$$

Let S be a semigroup. The *semigroup algebra* $\ell^1(S)$ is the completion in the ℓ^1-norm of the algebra $\mathbb{C}S$. It is the *Banach algebra generated by the semigroup*. The *convolution product* $\star$ on $\ell^1(S)$ is uniquely defined by requiring that $\delta_s \star \delta_t = \delta_{st} \ (s, t \in S)$. We identify $\ell^1(S)^\flat$ with $\ell^1(S^\flat)$, where $S^\flat$ is the semigroup formed by adjoining an identity to S. These Banach algebras have been studied by many authors. A recent exposition is the memoir [3].

2.3.1 Cancellativity

We shall use the following notation introduced by Grønbæk in [9]. For $s, t \in S$ we define the sets

$$[st^{-1}] = \{u \in S : ut = s\} \quad \text{and} \quad [t^{-1}s] = \{u \in S : tu = s\} .$$

Let S be a semigroup. An element $t \in S$ is *left cancellable* if $u = v$ whenever $tu = tv$. Equivalently we require that $|[t^{-1}s]| \leq 1 \ (s \in S)$. *Right cancellable* elements are defined similarly. The semigroup S is *cancellative* if each element is both left and right cancellative. The semigroup S is *weakly left* (respectively, *right*) *cancellative* if $[t^{-1}s]$ (respectively, $[st^{-1}]$) is finite for each $s, t \in S$, and *weakly cancellative* if it is both weakly left cancellative and weakly right cancellative.

Lemma 2.4. *Let S be a semigroup such that the Banach algebra $\ell^1(S)$ has a left identity. Suppose that S contains a right cancellable element. Then:*

(i) *S has a left identity e_S;*

(ii) *for each right cancellable element t we have $te_S = t$.*

Proof. (i) Let $e_A = \sum_{s\in S} e_s\delta_s$ be a left identity for $\ell^1(S)$, and let t be a right cancellable element, so that $|[tt^{-1}]| \leq 1$. We have

$$\delta_t = \sum_{s\in S} e_s\delta_{st} = \sum_{r\in S}\left(\sum_{s\in[rt^{-1}]} e_s\right)\delta_r\,.$$

Hence $\sum_{s\in[tt^{-1}]} e_s = 1$ and in particular the set $[tt^{-1}]$ is non-empty, say $[tt^{-1}] = \{u\}$. Then

$$\delta_u \star \delta_t = \delta_t = e_A \star \delta_t\,.$$

Since t is right cancellable, it follows that $\delta_u = e_A$, and so u is a left identity for S.

(ii) Let t be a right cancellable element and e_S a left identity for S. Then $t^2 = t(e_St) = (te_S)t$, which implies that $t = te_S$. □

2.3.2 Module actions

Let S be a semigroup, and let $E \in \ell^1(S)$-**mod** or $E \in$ **mod**-$\ell^1(S)$. We shall use the following more compact notation for the module actions

$$t\cdot x = \delta_t\cdot x,\quad x\cdot t = x\cdot\delta_t \quad (x\in E,\, t\in S)\,.$$

The standard actions of $\ell^1(S)$ on $\ell^1(S)$ are given by

$$(t\cdot a)(s) = \sum_{u\in[t^{-1}s]} a(u),\quad (a\cdot t)(s) = \sum_{u\in[st^{-1}]} a(u) \quad (s,t\in S,\, a\in\ell^1(S))\,,$$

where we define the sum over an empty set to be zero. The dual actions of $\ell^1(S)$ on $\ell^\infty(S)$ are given by

$$(t\cdot\lambda)(s) = \lambda(st),\quad (\lambda\cdot t)(s) = \lambda(ts) \quad (s,t\in S,\, \lambda\in\ell^\infty(S))\,.$$

For $s \in S$, the indicator function of the set $\{s\}$ will be denoted by δ_s when considered as an element of $\ell^1(S)$, and by λ_s when considered as an element of $\ell^\infty(S)$. This notation implies that the left module actions satisfy

$$t\cdot\delta_s = \delta_{ts} \quad\text{and}\quad t\cdot\lambda_s = \chi_{[st^{-1}]} \quad (t\in S)\,.$$

Proposition 2.5 ([3, Theorem 4.6])**.** *Let S be a semigroup. Then $c_0(S)$ is a left [right] $\ell^1(S)$-submodule of $\ell^\infty(S)$ if and only if S is weakly right [left] cancellative.* □

3 Some general results on injective modules

In this section we prove some basic intrinsic properties of injective modules. We begin with a generalization of [4, Proposition 1.8].

Proposition 3.1. *Let A be a Banach algebra, and let $E \in$ **mod**-A be injective. Suppose that there exists $Q \in \mathcal{B}(A, E)$ and a subset $S \subset A$ such that*

$$Q(ba) = Q(b) \cdot a \quad (b \in S,\, a \in A)\,.$$

Then there is an element $x \in E$ with

$$Q(b) = x \cdot b \quad (b \in S)\,.$$

Proof. There exists a morphism $\rho \in \mathcal{B}_A(\mathcal{B}(A^\flat, E), E)$ such that $\rho \circ \Pi = I_E$. Extend Q to $A^\flat$ by setting $Q(e^\flat) = 0$. Then $Q \in \mathcal{B}(A^\flat, E)$ and

$$Q \cdot b = \Pi(Q(b)) \quad (b \in S)\,.$$

Set $x = \rho(Q) \in E$. Then we have

$$Q(b) = (\rho \circ \Pi)(Q(b)) = \rho(Q \cdot b) = x \cdot b \quad (b \in S)\,,$$

as required. □

Corollary 3.2. *Let A be a Banach algebra, let I be a complemented ideal in A, and let $E \in$ **mod**-A be injective. Then the map $j : E \to \mathcal{B}_A(I, E)$ given by*

$$j(x) : a \mapsto x \cdot a : I \to E$$

is a Banach A-module epimorphism with kernel $E^{\perp I}$.

Proof. Take $T \in \mathcal{B}_A(I, E)$, and set $Q = T \circ P$, where $P : A^\flat \to I$ is a projection. Then $Q \in \mathcal{B}(A^\flat, E)$ and satisfies $Q(ba) = Q(b) \cdot a$ $(b \in I, a \in A)$. By Proposition 3.1, there exists $x \in E$ such that $T(b) = Q(b) = x \cdot b$ $(b \in I)$, i.e., $T = j(x)$.

The rest is clear. □

Corollary 3.3. *Let A be a Banach algebra.*

(i) *Let A be a subalgebra of a Banach algebra B. Suppose that B is injective in* **mod**-A. *Then there exists $p \in B$ with $pa = a$ $(a \in A)$.*

(ii) *Let I be a closed right ideal in A. Suppose that A/I is injective in* **mod**-A. *Then I has a left modular identity.*

(iii) *Let I be a closed, complemented right ideal in A. Suppose that I is injective in* **mod**-A. *Then there exists $p \in I$ with $pa = a$ $(a \in I)$.*

(iv) *Let I be a closed, complemented right ideal in A. Suppose that I'' is injective in* **mod**-A. *Then I has a bounded left approximate identity.*

Proof. These all follow from Proposition 3.1 by choosing specific maps Q.

(i) Take $Q : A \to B$ to be the inclusion map.

(ii) Take $Q : A \to A/I$ to be the quotient map.

(iii) Take $Q : A \to I$ to be a projection.

(iv) Take the map $Q : A \to I''$ to be a projection onto I followed by the natural embedding into I''. Then there exists $\Phi \in I''$ with $\Phi \cdot a = a$ $(a \in I)$. The result now follows by a standard argument involving the weak-$*$ topology on I'', see [2, Proposition 2.9.14(iii)]. □

Proposition 3.4. *Let A be a Banach algebra, and let $E \in$ **mod**-A be injective. Suppose that there exists $a_0 \in A \setminus \{0\}$ with $a_0 A = 0$. Then $E^\perp = E \cdot a_0$. In particular, the subspace $E \cdot a_0$ is closed.*

Proof. The inclusion $E \cdot a_0 \subset E^\perp$ is clear.

Let $x \in E^\perp$. Take $\mu \in (A^\flat)'$ with $\langle a_0, \mu \rangle = 1$, and let $T \in \mathcal{B}(A^\flat, E)$ be the rank-1 operator given by

$$T(a) = \langle a, \mu \rangle \, x \quad (a \in A^\flat)\,.$$

Let $P_A, P_{\mathbb{C}e^\flat} \in \mathcal{B}(A^\flat)$ be projections on to the subspaces A and $\mathbb{C}e^\flat$, respectively. Then we have $(\Pi x) \circ P_{\mathbb{C}e^\flat} = (T \cdot a_0) \circ P_{\mathbb{C}e^\flat}$.

Clearly, for any $S \in \mathcal{B}(A^\flat, E)$, we have

$$\rho(S) = \rho(S \circ P_A) + \rho(S \circ P_{\mathbb{C}e^\flat})\,.$$

Combining this with the fact that $\rho((\Pi x) \circ P_A) = \rho((T \cdot a_0) \circ P_A) = 0$, we have

$$x = \rho((\Pi x) \circ P_{\mathbb{C}e^\flat}) = \rho((T \cdot a_0) \circ P_{\mathbb{C}e^\flat}) = \rho(T \cdot a_0) = \rho(T) \cdot a_0\,.$$

Therefore $E^\perp = E \cdot a_0$. □

Example 3.5 ([16, Example 4.4])**.** Let X be a Banach space with dim $X \geq 2$, and take $\varphi \in X' \setminus \{0\}$. Define a product on X by

$$ab = \varphi(a)b \quad (a, b \in X)\,.$$

With this product X is a Banach algebra which, following [16], we denote by $A_\varphi(X)$. By [16], $A_\varphi(X)$ is a biprojective Banach algebra. Since $A_\varphi(X)$ does not have a right identity, $A_\varphi(X)$ is not injective in $A_\varphi(X)$-**mod**. The Banach algebra $A_\varphi(X)$ is faithful in $A_\varphi(X)$-**mod**, but not faithful in **mod**-$A_\varphi(X)$.

Proposition 3.6. *The Banach algebra $A_\varphi(X)$ is injective in* **mod**-$A_\varphi(X)$ *if and only if*

$$\dim X = 2\,.$$

Proof. Set $A = A_\varphi(X)$.

Suppose that A is injective in **mod**-A. Choose $a_0 \in \ker \varphi \setminus \{0\}$. By Proposition 3.4 we have

$$\ker \varphi = A^\perp = Aa_0 = \mathbb{C}a_0\,.$$

By the rank-nullity theorem, dim $X = 2$.

Conversely, suppose that dim $X = 2$. Let $\{a_0, a_1\}$ be a basis of X with $\langle a_0, \varphi \rangle = 0$ and $\langle a_1, \varphi \rangle = 1$. Take $\lambda \in X'$ with $\langle a_0, \lambda \rangle = 1$ and $\langle a_1, \lambda \rangle = 0$. We define a map $\rho : \mathcal{B}(A^\flat, A) \to A$ by

$$\rho(T) = \langle Ta_0, \lambda \rangle \, a_1 - \langle Ta_1, \lambda \rangle \, a_0 + \left\langle Te^\flat, \lambda \right\rangle a_0 \quad (T \in \mathcal{B}(A^\flat, A))\,.$$

This is a right A-module morphism with $\rho \circ \Pi = I_A$. The easiest way to see this is to check that $\rho(T \cdot a_0) = \rho(T) \cdot a_0$, $\rho(T \cdot a_1) = \rho(T) \cdot a_1$ $(T \in \mathcal{B}(A^\flat, A))$, and $\rho(\Pi a_0) = a_0$, $\rho(\Pi a_1) = a_1$. Therefore A is injective in **mod**-A. □

4 Amenability and injectivity

In this section way show how the injectivity of certain modules over a semigroup algebra implies that the underlying semigroup must be amenable. This is a generalization of the argument in [4, §4]. In contrast to the group case, for a general semigroup, amenability is only part of the story.

4.1 General results

Definition 4.1. Let A be a Banach algebra, and let $E \in \mathbf{mod}\text{-}A$. Then an element $\lambda \in E' \setminus \{0\}$ is an *augmentation-invariant* functional if there exists a character φ on A, with $a \cdot \lambda = \varphi(a)\lambda$ for each $a \in A$. The triple (E, λ, φ) is an *augmentation-invariant* Banach right A-module.

Example 4.2. (i) Let A be a Banach algebra with a character φ, and let I be a closed left ideal of A with $I \not\subset \ker \varphi$. Then $(I, \varphi|I, \varphi)$ is an augmentation-invariant right A-module.

(ii) If $E \in \mathbf{mod}\text{-}A$ is augmentation-invariant, then so is E''.

Lemma 4.3. *Let A be a Banach algebra, let I be a complemented left ideal of A, and let (E, λ, φ) be an augmentation-invariant Banach right A-module with $I \not\subset \ker \varphi$. Suppose that E is injective in* $\mathbf{mod}\text{-}A$*. Then there exists $\Lambda \in \mathcal{B}(I, E)'$ such that:*

(i) $a \cdot \Lambda = \varphi(a)\Lambda$ *for each* $a \in A$*;*

(ii) $\langle \Pi(x), \Lambda \rangle = \langle x, \lambda \rangle$ *for each* $x \in E$*.*

Proof. Since E is injective there is a right A-module morphism $\rho : \mathcal{B}(A^\flat, E) \to E$ with $\rho \circ \Pi = I_E$. Set $\Lambda_0 = \rho'(\lambda) \in \mathcal{B}(A^\flat, E)'$. Since ρ' is a left A-module morphism, we have $a \cdot \Lambda_0 = \varphi(a)\Lambda_0 \ (a \in A)$.

Take $T \in \mathcal{B}(A^\flat, E)$ such that $T|I = 0$. Pick $a \in I$ with $\varphi(a) = 1$. Then $T \cdot a = 0$, and

$$0 = \langle T \cdot a, \Lambda_0 \rangle = \langle T, \Lambda_0 \rangle \ .$$

Now take $T \in \mathcal{B}(I, E)$. We can extend T to $\widetilde{T} \in \mathcal{B}(A^\flat, E)$ by setting $\widetilde{T} = T \circ P$, where $P : A^\flat \to I$ is a projection. Set

$$\langle T, \Lambda \rangle := \left\langle \widetilde{T}, \Lambda_0 \right\rangle \quad (T \in \mathcal{B}(I, E))\,.$$

Since $\left(\widetilde{T \cdot a} - \widetilde{T} \cdot a\right)|I = 0 \ (a \in A)$, it follows that $a \cdot \Lambda = \varphi(a)\Lambda \ (a \in A)$. Similarly, since $\left(\widetilde{\Pi(x)} - \Pi(x)\right)|I = 0 \ (x \in E)$, it follows that $\langle \Pi(x), \Lambda \rangle = \langle x, \lambda \rangle$. □

In the following theorem we set $\widetilde{\pi} = \Pi' : \mathcal{B}(I, E)' = (I \widehat{\otimes} F)'' \to E' = F''$. If F is a left A-submodule of F'', then $\widetilde{\pi}|I \widehat{\otimes} F \subset F$, and we can replace $\widetilde{\pi}$ by π.

Theorem 4.4. *Let A be a Banach algebra, let I be a complemented left ideal of A, and let (E, λ, φ) be an augmentation-invariant Banach right A-module with $I \not\subset \ker \varphi$. Suppose that E is the dual of a Banach space F, and that E is injective in* $\mathbf{mod}\text{-}A$*. Then there exists a bounded net $(v_\alpha) \subset I \widehat{\otimes} F$ such that:*

(i) $\lim_\alpha \|a \cdot v_\alpha - \varphi(a)v_\alpha\|_\pi = 0$ *for each* $a \in A$*;*

(ii) $\lim_\alpha \langle x, \widetilde{\pi}(v_\alpha) \rangle = \langle x, \lambda \rangle$ *for each* $x \in E$*.*

Proof. Set $X = I \widehat{\otimes} F$, and let $\sigma = \sigma(X, X')$ be the weak topology on X.

First, a net (u_α) is indexed by the family of all finite subsets of $\mathcal{B}(I, E)$, with the ordering specified by inclusion. For each such $\alpha = \{T_1, \ldots, T_k\}$, choose $u_\alpha \in X$ such that $\langle T_i, u_\alpha \rangle = \langle T_i, \Lambda \rangle \ (i = 1, \ldots, k)$, where $\Lambda \in X''$ was specified in Lemma 4.3.

For each $a \in A$ and $T \in \mathcal{B}(I, E)$, we have

$$\langle T, a \cdot u_\alpha \rangle = \langle T \cdot a, u_\alpha \rangle = \langle T \cdot a, \Lambda \rangle = \varphi(a) \langle T, \Lambda \rangle = \varphi(a) \langle T, u_\alpha \rangle ,$$

for each sufficiently large α, and so $\lim_\alpha (a \cdot u_\alpha - \varphi(a) u_\alpha) = 0$ in (X, σ).

Also for each $x \in E$, we have

$$\langle x, \widetilde{\pi}(u_\alpha) \rangle = \langle \Pi(x), u_\alpha \rangle = \langle \Pi(x), \Lambda \rangle = \langle x, \lambda \rangle ,$$

for each sufficiently large α, and so $\lim_\alpha \langle \Pi(x), u_\alpha \rangle - \langle x, \lambda \rangle = 0$.

Let $\{a_1, \ldots, a_k\}$ and $\{x_1, \ldots, x_\ell\}$ be finite subsets of A and E, respectively, and let $\varepsilon > 0$. Let

$$C = C(\{x_1, \ldots, x_\ell\}, \varepsilon) = \{z \in X : |\langle \Pi(x_i), z \rangle - \langle x_i, \lambda \rangle| < \varepsilon \ (i = 1, \ldots, \ell)\} ,$$

and consider the Banach space $Y = X_1 \oplus \cdots \oplus X_k$, where each $X_i = X \ \ (i = 1, \ldots, k)$ and we are taking the ℓ^1-sum. Also consider the linear operator

$$W : z \mapsto (a_1 \cdot z - \varphi(a_1) z, \ldots, a_k \cdot z - \varphi(a_k) z), \quad X \to Y .$$

The set C is convex in X, and so $W(C)$ is convex in Y. We have shown that 0 belongs to the $\sigma(Y, Y')$-closure of $W(C)$ in Y. By Mazur's theorem, it follows that 0 belongs to the $\|\cdot\|$-closure of $W(C)$ in Y. The existence of the required net (v_α) follows. □

4.2 The Banach algebras $L^1(G)$ and $M(G)$

Let G be a locally compact group, and let $M(G)$ denote the measure algebra on G. There is always one character on $M(G)$; this is the *augmentation character* φ_G, defined by

$$\varphi_G(\mu) = \mu(G) \quad (\mu \in M(G)) .$$

The restriction of φ_G to $L^1(G)$ regarded as a closed ideal in $M(G)$ has the form

$$\varphi_G(f) = \int_G f(s) \, \mathrm{d}m(s) \quad (f \in L^1(G)) .$$

Let G be a locally compact group. We set

$$P(G) = \left\{ f \in L^1(G) : f \geq 0, \, \|f\| = 1 \right\} .$$

We shall use the following characterization of amenability, known as *Reiter's condition.*

Proposition 4.5 ([13, Proposition (0.8)]). *Let G be a locally compact group. Then G is amenable if and only if there is a net $(f_\alpha) \subset P(G)$ such that* $\lim \|t \cdot f_\alpha - f_\alpha\| = 0$ *for each $t \in G$.* □

Theorem 4.6. *Let G be a locally compact group, and let (E, λ, φ_G) be an augmentation-invariant Banach right $M(G)$-module. Suppose that E is a dual space, and that E is injective in* **mod**-$M(G)$. *Then G is amenable.*

Proof. Let E have a Banach space predual F. Set $A = M(G)$ and $I = L^1(G)$. Let $(v_\alpha) \subset I \widehat{\otimes} F$ be the net given by Theorem 4.4. Take $x \in E_{[1]}$ with $\langle x, \lambda \rangle = 1$. We have

$$\|v_\alpha\|_\pi \geq \|\widetilde{\pi}(v_\alpha)\|_{F''} \geq |\langle x, \widetilde{\pi}(v_\alpha)\rangle| \geq 1/2$$

for large enough α. Hence by passing to a subnet we may suppose that, for each α, $\|v_\alpha\|_\pi \geq 1/2$. We use the identification $I \widehat{\otimes} F = L^1(G, F)$ to define a net (k_α) in I by

$$k_\alpha(s) = \frac{\|v_\alpha(s)\|_F}{\|v_\alpha\|_\pi} \quad (s \in G)\,.$$

Then $k_\alpha \geq 0$, and

$$\|k_\alpha\|_1 = \int_G k_\alpha(s)\,\mathrm{d}m(s) = \int_G \frac{\|v_\alpha(s)\|_F}{\|v_\alpha\|_\pi}\,\mathrm{d}m(s) = 1\,.$$

Now take $t \in G$. We have

$$\|t \cdot k_\alpha - k_\alpha\|_1 \leq \frac{1}{\|v_\alpha\|_\pi} \int_G \left\|v_\alpha(t^{-1}s) - v_\alpha(s)\right\|_F \,\mathrm{d}m(s) \leq 2\,\|t \cdot v_\alpha - v_\alpha\|_\pi\,.$$

Hence $\lim_\alpha \|t \cdot k_\alpha - k_\alpha\|_1 = 0$. Therefore by Proposition 4.5, G is amenable. □

The same result holds under the hypothesis that E is injective in **mod**-$L^1(G)$. This combines with Johnson's theorem and Proposition 2.2 to give the following result, which was proved under additional hypothesis in [4, Theorem 4.6].

Theorem 4.7. *Let G be a locally compact group, and let (E, λ, φ_G) be an augmentation-invariant Banach right $L^1(G)$-module. Suppose that E is a dual space. Then E is injective in* **mod**-*$L^1(G)$ if and only if G is amenable.* □

These theorems apply to the $L^1(G)$ or $M(G)$-modules; $L^1(G)$, $M(G)$ and their second duals. Some further results about modules over $M(G)$ are contained in [14].

4.3 The Banach algebra $\ell^1(S)$

Theorem 4.8. *Let S be a semigroup such that $\ell^1(S)$ is injective in* **mod**-*$\ell^1(S)$. Then S is a left-amenable semigroup and $\ell^1(S)$ has a left identity.*

Proof. That S is left-amenable follows in the same way as the proof of Theorem 4.6. That $\ell^1(S)$ has a left identity is Corollary 3.3(i). □

Let A be a Banach algebra with a left identity e_A. Then e_A is an identity for A if and only if $\{a \in A : aA = 0\} = \{0\}$ i.e., A is a faithful right A-module.

Lemma 4.9. *Let S be a semigroup such that $\ell^1(S)$ is injective in* **mod**-*$\ell^1(S)$. Let $t \in S$ be a left cancellable element. Then there exists $a_t \in \ell^1(S)$ with*

$$a_t \star \delta_t \star e_A = e_A$$

for each left identity $e_A \in \ell^1(S)$.

Proof. Set $A = \ell^1(S)$. There is a map

$$T : tS \to S : ts \mapsto s$$

which extends to a bounded linear operator $T : \ell^1(tS) \to A$. Set $U = T \circ P$, where $P : A \to \ell^1(tS)$ is a projection. Then U satisfies

$$U(b \star a) = U(b) \star a \quad (b \in \ell^1(tS),\, a \in A)\,.$$

By Proposition 3.1, there exists $a_t \in A$ with $U(b) = a_t \star b$ $(b \in \ell^1(tS))$. In particular, $e_A = U(\delta_t \star e_A) = a_t \star \delta_t \star e_A$. ☐

Theorem 4.10. *Let S be a cancellative semigroup. Then $\ell^1(S)$ is injective in* **mod**-$\ell^1(S)$ *if and only if S is an amenable group.*

Proof. Sufficiency is obvious; we shall prove necessity.

Suppose that $\ell^1(S)$ is an injective right module. By Theorem 4.8, the Banach algebra $\ell^1(S)$ has a left identity. By Lemma 2.4, S has an identity e_S. By Lemma 4.9, for each $s \in S$, there exists $a_s \in \ell^1(S)$ with $a_s \star \delta_s = \delta_{e_S}$. It follows that there is an element $s^{-1} \in S$ with $s^{-1}s = e_S$. From the equation $ss^{-1}s = e_S s$ and right cancellativity we have $ss^{-1} = e_S$. Therefore S is a group. It is amenable by Theorem 4.8. ☐

Corollary 4.11. *The Banach algebra $\ell^1(\mathbb{N})$ is not injective in* **mod**-$\ell^1(\mathbb{N})$*, equivalently $c_0(\mathbb{N})$ is not flat in* $\ell^1(\mathbb{N})$-**mod**. ☐

Example 4.12. Let S be the *right-zero semigroup*. The product is given by

$$st = t \quad (s, t \in S)\,.$$

The Banach algebra $\ell^1(S)$ belongs to the class of Banach algebras of the form $A_\varphi(X)$ defined in Example 3.5. By Proposition 3.6, $\ell^1(S)$ is right injective if and only if $|S| = 2$. Since in this case $\ell^1(S)$ is finite dimensional, this is equivalent to the predual module $c_0(S) = \ell^\infty(S)$ being projective in $\ell^1(S)$-**mod**.

Let S_2 be the right-zero semigroup on 2 elements. We note that $\ell^1(S_2)$ is not amenable. Indeed S_2 is left-amenable but not right-amenable; further, S_2 has a left identity, but $\ell^1(S_2)$ does not have a right identity. Hence the plausible conjecture that $\ell^1(S)$ is injective in **mod**-$\ell^1(S)$ only if $\ell^1(S)$ is amenable is false.

5 Flatness of the predual module $c_0(S)$

Let S be a weakly right cancellative semigroup. Then, by Proposition 2.5 $c_0(S) \in \ell^1(S)$-**mod**, and we can identify the dual right $\ell^1(S)$-module $c_0(S)'$ with $\ell^1(S)$. Hence for weakly right cancellative semigroups, injectivity of $\ell^1(S)$ in **mod**-$\ell^1(S)$ is the same as flatness of $c_0(S)$ in $\ell^1(S)$-**mod**. In this section we shall study this problem for the class of weakly right cancellative semigroups.

5.1 A necessary combinatorial condition

For the next two lemmas, we suppose that S is a semigroup, and that E is a Banach space. For a subset $T \subset S$, we set $X_T = \ell^1(T) \widehat{\otimes} E$. For an element $t \in S$, and $F \in \ell^1(S)$-**mod**, we set ${}^{t\perp}F = {}^{\{\delta_t\}\perp}F$.

Lemma 5.1. *For each $t \in S$, we have*

$$({}^{t\perp}X_S)^{\,0} = X'_S \cdot t\,.$$

Proof. Let $t \in S$. The inclusion $X'_S \cdot t \subset ({}^{t\perp}X_S)^{\,0}$ is clear.

Take $\lambda \in ({}^{t\perp}X_S)^{\,0}$ and $u, v \in S$ with $tu = tv$. Then for each $x \in E$ we have

$$\delta_u \otimes x - \delta_v \otimes x \in {}^{t\perp}X_S\,.$$

Hence, under the identification $X'_S = \ell^{\infty}(S, E')$, we have

$$0 = \langle \delta_u \otimes x - \delta_v \otimes x, \lambda\rangle = \langle x, \lambda(u) - \lambda(v)\rangle \quad (x \in E)\,.$$

It follows that $\lambda(u) = \lambda(v)$. Hence, for each $s \in S$, λ is constant on the set $[t^{-1}s]$. Therefore we can define

$$\varphi(s) = \begin{cases} \lambda(u), & \text{if there exists } u \in [t^{-1}s]\,, \\ 0, & \text{if } [t^{-1}s] = \emptyset\,, \end{cases} \qquad (s \in S)\,.$$

Then $\varphi \in X'_S$ and $\lambda = \varphi \cdot t$. $\square$

For a subset $T \subset S$ and an element $t \in S$, we define the following subset of T:

$$G(T,t) = \bigcup_{\{s \in S : |[t^{-1}s] \cap T| \geq 2\}} [t^{-1}s] \cap T = \{u \in T : \exists v \in T, v \neq u, tu = tv\}\,.$$

The complement of $G(T,t)$ in T is perhaps simpler to describe: we have

$$T \setminus G(T,t) = \{s \in T : [t^{-1}(ts)] \cap T = \{s\}\}\,.$$

For example, if t is a left cancellable element, then $G(T,t) = \emptyset$.

For a subset $T \subset S$, we identify X_T with the closed, complemented subspace of X_S consisting of functions on S whose support is contained in T. We can then identify X''_T with X_T^{00} in X''_S.

Lemma 5.2. *For each $t \in S$ and $T \subset S$, we have:*

(i) $t \cdot X''_S \subset (X_{tS})''$;

(ii) ${}^{t\perp}(X''_T) \subset X''_{G(T,t)}$.

Proof. (i) Let $t \in S$ and $\varphi \in (X_{tS})^{\,0}$. Then $\varphi \cdot t = 0$, and so for each $\Lambda \in X''_S$, we have

$$\langle \varphi, t \cdot \Lambda\rangle = \langle \varphi \cdot t, \Lambda\rangle = 0\,.$$

Therefore $t \cdot \Lambda \in X_{tS}^{00} = X''_{tS}$.

(ii) Let $t \in S$ and $T \subset S$. Take $z = \sum_{s \in S} \delta_s \otimes x_s \in {}^{t\perp}X_T$. The equation $t \cdot z = 0$ gives

$$\sum_{u \in [t^{-1}s] \cap T} x_u = 0 \quad (s \in S)\,.$$

Take $u \in \operatorname{supp} z$. Then $u \in [t^{-1}(tu)] \cap T$. Hence $|[t^{-1}(tu)] \cap T| \geq 2$, and $z \in X_{G(T,t)}$. We have proved that ${}^{t\perp}X_T \subset X_{G(T,t)}$. Now by Lemma 5.1 we have

$$(X_{G(T,t)})^{\,0} \subset ({}^{t\perp}X_T)^{\,0} = X'_T \cdot t\,.$$

Hence ${}^{t\perp}(X''_T) = (X'_T \cdot t)^{\,0} \subset (X_{G(T,t)})^{\,00} = X''_{G(T,t)}$. $\square$

Theorem 5.3. *Let S be a weakly right cancellative semigroup such that, for each $N \in \mathbb{N}$, there exist elements $(s_1, r_1, t_1), \ldots, (s_N, r_N, t_N)$ in $S \times S \times S$ with the following properties:*

(i) $s_n \in Sr_n \setminus St_nr_n \ (n \in \mathbb{N}_N)$;

(ii) *the sets* $[s_1r_1^{-1}], \ldots, [s_Nr_N^{-1}]$ *are pairwise disjoint;*

(iii) *the sets* $G(r_1S^\flat, t_1), \ldots, G(r_NS^\flat, t_N)$ *are pairwise disjoint.*

*Then $c_0(S)$ is not flat in $\ell^1(S)$-**mod**.*

Proof. We set $E = c_0(S)$, and for a subset $T \subset S^\flat$, we set $A_T = \ell^1(T)$.

Assume towards a contradiction that E is flat in $\ell^1(S)$-**mod**. Then there exists a left $\ell^1(S)$-module morphism $\rho : E \to (A_{S^\flat} \widehat{\otimes} E)''$ with $\pi'' \circ \rho = i_E$. Fix $N \in \mathbb{N}$, and let $(s_1, r_1, t_1), \ldots, (s_N, r_N, t_N) \in S \times S \times S$ be the elements given by the hypothesis. For each $n \in \mathbb{N}_N$, set

$$x_n = r_n \cdot \lambda_{s_n} = \chi_{[s_nr_n^{-1}]} \,.$$

By Lemma 5.2(i) $\rho(x_n) \in (A_{r_nS^\flat} \widehat{\otimes} E)''$. Since $s_n \notin St_nr_n$, we have

$$t_n \cdot \rho(x_n) = \rho(t_n \cdot x_n) = \rho(\chi_{[s_n(t_nr_n)^{-1}]}) = 0\,,$$

Hence by Lemma 5.2(ii) $\rho(x_n) \in (A_{G(r_nS^\flat, t_n)} \widehat{\otimes} E)''$.

Set $\Phi = \rho\left(\sum_{n=1}^N x_n\right) \in (A_{S^\flat} \widehat{\otimes} E)''$. By (ii) $\left\|\sum_{n=1}^N x_n\right\|_\infty = 1$. Hence there is a net (z_α) in $(A_{S^\flat} \widehat{\otimes} E)_{[\|\rho\|]}$ such that $\lim_\alpha z_\alpha = \Phi$ in the weak-$*$ topology. For each $n \in \mathbb{N}_N$, let $P_n : A_{S^\flat} \widehat{\otimes} E \to A_{G(r_nS^\flat, t_n)} \widehat{\otimes} E$ be a projection. By (iii) we have $P_n''(\rho(x_m)) = 0 \ (n \neq m)$. Hence $\lim_\alpha P_n''(z_\alpha) = P_n''(\Phi) = \rho(x_n)$ in the weak-$*$ topology.

For each $i \in \mathbb{N}_N$, by (i) we can pick $u_n \in [s_nr_n^{-1}]$. For large enough α, we have

$$|\langle \Pi(\delta_{u_n}), P_n(z_\alpha)\rangle - \langle \Pi(\delta_{u_n}), \rho(x_n)\rangle| < 1/2\,.$$

Now for each $n \in \mathbb{N}_N$, we have

$$\langle \Pi(\delta_{u_n}), \rho(x_n)\rangle = \langle \delta_{u_n}, \pi'' \circ \rho(x_n)\rangle = \langle x_n, \delta_{u_n}\rangle = 1\,.$$

Hence, for large enough α, we have

$$|\langle \Pi(\delta_{u_n}), P_n(z_\alpha)\rangle| > 1/2\,,$$

and so $\|P_n(z_\alpha)\|_\pi \geq 1/2$. But now using (iii), for sufficiently large α,

$$\|\rho\| \geq \|z_\alpha\|_\pi \geq \sum_{n=1}^N \|P_n(z_\alpha)\|_\pi \geq N/2\,.$$

This holds for each $N \in \mathbb{N}$, the required contradiction. □

5.2 A special case

Here we describe a special case of the condition in Theorem 5.3, which is easier to apply to certain examples. The key is the following description of the sets $G(qS, p)$.

Lemma 5.4. *Let S be a semigroup, and let $p, q \in E(S)$ with $p < q$. Then $pS \subset qS$ and*

$$G(qS, p) = (qS \setminus pS) \cup p(qS \setminus pS)$$

is a disjoint union of sets.

Proof. It is clear that $pS \subset qS$ and $(qS \setminus pS) \cap p(qS \setminus pS) = \emptyset$.

Let $\pi_1 : S \times S \to S$ be the projection onto the first coordinate. Consider the sets

$$\mathcal{G} = \{(u, v) \in qS \times qS : u \neq v, pu = pv\}$$

and

$$\mathcal{F} = \{(u, pu) : u \in qS \setminus pS\} \cup \{(pu, u) : u \in qS \setminus pS\}.$$

We make the identifications

$$\pi_1(\mathcal{G}) = G(qS, p) \quad \text{and} \quad \pi_1(\mathcal{F}) = (qS \setminus pS) \cup p(qS \setminus pS).$$

Since $\mathcal{F} \subset \mathcal{G}$ we have $\pi_1(\mathcal{F}) \subset \pi_1(\mathcal{G})$. Now take $u \in \pi_1(\mathcal{G})$ with $u = \pi_1((u, v))$ for some $(u, v) \in \mathcal{G}$. If $u \notin pS$, then $(u, pu) \in \mathcal{F}$. If $u \in pS$, then $v \in qS \setminus pS$ and $(u, v) = (pv, v) \in \mathcal{F}$. In either case $u \in \pi_1(\mathcal{F})$. □

Theorem 5.5. *Let S be a weakly right cancellative semigroup. Suppose that there exists an infinite chain of idempotents*

$$r_1 > s_1 > t_1 > \cdots > r_n > s_n > t_n > r_{n+1} > s_{n+1} > t_{n+1} > \cdots$$

*such that for each $n \in \mathbb{N}$, $t_n(r_nS \setminus t_nS) \cap r_{n+1}S = \emptyset$. Then $c_0(S)$ is not flat in $\ell^1(S)$-**mod**.*

Proof. We shall apply Theorem 5.3; we verify clauses (i)-(iii).

Clearly $s_n \in Sr_n \setminus St_n = Sr_n \setminus St_nr_n$ $(n \in \mathbb{N})$, so that clause (i) holds.

Take $n < m$. Assume towards a contradiction that there exists $u \in [s_nr_n^{-1}] \cap [s_mr_m^{-1}]$. Then $ur_n = s_n$ and $ur_m = s_m$. Multiplying the first of these equations on the right by r_m gives $ur_m = r_m$. Hence $r_m = s_m$, which is a contradiction. Therefore clause (ii) holds.

For each $l \leq m$ we have

$$t_kS \supset r_mS.$$

Hence we have

$$(r_nS \setminus t_nS) \cap G(r_mS, t_m) \subset (S \setminus t_nS) \cap r_mS = \emptyset.$$

Using Lemma 5.4, we have

$$\begin{aligned} G(r_nS, t_n) \cap G(r_mS, t_m) &= t_n(r_nS \setminus t_nS) \cap G(r_mS, t_m) \\ &\subset t_n(r_nS \setminus t_nS) \cap r_mS \\ &\subset t_n(r_nS \setminus t_nS) \cap r_{n+1}S \\ &= \emptyset \quad \text{(by hypothesis)}. \end{aligned}$$

Therefore condition (iii) of Theorem 5.3 also holds, and hence $c_0(S)$ is not flat in $\ell^1(S)$-**mod**. □

Example 5.6. We give some examples of semigroups which satisfy the hypothesis of Theorem 5.5.

(i) Let $\mathbb{N}_\vee = (\mathbb{N}, \max)$. The canonical partial order on $\mathbb{N}_\vee$ is the reverse of the natural order on $\mathbb{N}$. We set

$$r_n = 3n - 2, \quad s_n = 3n - 1, \quad t_n = 3n \quad (n \in \mathbb{N}).$$

For any $n \in S$, we have $n\mathbb{N}_\vee = [n, \infty)$. Hence for each $n \in \mathbb{N}$ we have $t_n(r_n\mathbb{N}_\vee \setminus t_n\mathbb{N}_\vee) = 3n[3n-2, 3n-1] = \{3n\}$, which is disjoint from the set $r_{n+1}\mathbb{N}_\vee = [3n+1, \infty)$. Therefore $c_0(\mathbb{N}_\vee)$ is not flat in $\ell^1(\mathbb{N}_\vee)$-**mod**.

(ii) Let B be the *bicyclic semigroup*. Then $B = \mathbb{N}_0 \times \mathbb{N}_0$ with the multiplication

$$(m,n)(p,q) = (m-n+\max\{n,p\}, q-p+\max\{n,p\}) \quad ((m,n),(p,q) \in B)\,.$$

We set

$$r_n = (3n-2, 3n-2), \quad s_n = (3n-1, 3n-1), \quad t_n = (3n, 3n) \quad (n \in \mathbb{N})\,.$$

For any $(m,n) \in B$, we have $(m,n)B = [m, \infty) \times \mathbb{N}_0$. Hence for each $n \in \mathbb{N}$ we have

$$t_n(r_nB \setminus t_nB) = (3n,3n)([1, 3n-1] \times \mathbb{N}_0) = \{3n\} \times \mathbb{N}_0\,,$$

which is disjoint from the set $r_{n+1}B = [3n+1, \infty) \times \mathbb{N}_0$. Therefore $c_0(B)$ is not flat in $\ell^1(B)$-**mod**.

The next example is 'at the opposite extreme' to those above. This semigroup does not satifty the hypothesis of Theorem 5.3 (and hence also of Theorem 5.5).

Example 5.7. Let X be a set, let $P_X = \mathcal{P}(X)$ be the power set of X, and let F_X be the set of all finite subsets of X. Then P_X is a semilattice with the multiplication

$$st = s \cup t \quad (s,t \in P_X)\,,$$

and F_X is a subsemilattice of P_X called the *free semilattice* over X. The empty set is the identity of P_X. For $s,t \in P_X$, we have

$$[st^{-1}] = [t^{-1}s] = \{u : ut = s\} = \begin{cases} \emptyset \text{ if } t \not\subset s \\ \{s \setminus u : u \subset t\} \text{ if } t \subset s \end{cases}.$$

For $t \subset s \in F_X$ we have $|[t^{-1}s]| = 2^{|t|}$, and hence F_X is weakly cancellative. For each $t \in P_X$, we have $[tt^{-1}] = \{u : u \subset t\}$. The canonical partial order on P_X is given by

$$s \le t \Longleftrightarrow t \subset s \quad (s,t \in P_X)\,.$$

Take $r,t \in P_X$ with $r \not\lesssim t$ i.e., $t \setminus r \neq \emptyset$. Take $u \in rP_X$. We can write $u = r \cup s$ where $s \cap r = \emptyset$. Set

$$v = \begin{cases} u \cup (t \setminus u) \text{ if } t \setminus u \neq \emptyset \\ u \setminus (s \cap t) \text{ if } s \cap t \neq \emptyset \end{cases}.$$

The condition $t \setminus r \neq \emptyset$ ensures that one of these cases must occur. Then $u \neq v$ and $t \cup u = t \cup v$. Hence $u \in G(rP_X, t)$ and $G(rP_X, t) = rP_X$. Hence for $r_1 \not\lesssim t_1$ and $r_2 \not\lesssim t_2$ we have $G(r_1P_X, t_1) \cap G(r_2P_X, t_2) = r_1r_2P_X$. Therefore Theorem 5.5 gives no information about the injectivity of the module $\ell^1(P_X)$.

It is proved in [14] that if X is an infinite set, then $\ell^1(P_X)$ and $\ell^1(F_X)$ are not right injective Banach algebras. The proof of this result involves a long technical combinatorial calculation and will be presented elsewhere.

6 Projectivity of the predual module $c_0(S)$

Again for a weakly right cancellative semigroup S we now consider when $c_0(S)$ has the stronger property of being projective in $\ell^1(S)$-**mod**.

6.1 Some technical 'smallness' results

Lemma 6.1. *Let S be an infinite, weakly right cancellative semigroup such that, for every finite set $F \subset S$, there exists $r \in S$ with $rS^\flat \cap F = \emptyset$. Suppose that $c_0(S)$ is projective in $\ell^1(S)$-**mod** with splitting morphism $\rho : c_0(S) \to \ell^1(S^\flat) \widehat{\otimes} c_0(S)$. Then for each $N \in \mathbb{N}$, there exist elements $x_1, \ldots, x_N$ in $c_0(S)$ and a partition $\{F_1, \ldots, F_N\}$ of S with the properties:*

(i) $\left\| \sum_{i=1}^N x_i \right\|_\infty = 1$,

(ii) $\|\rho(x_i)\|_\pi \geq 1$ *for each* $i \in \mathbb{N}_N$, *and,*

(iii) $\|\chi_{F_i}\rho(x_i) - \rho(x_i)\|_\pi < 1/3^i$ *for each* $i \in \mathbb{N}_N$.

Proof. We set $E = c_0(S)$, and for a subset $T \subset S$, we set $A_T = \ell^1(T)$.

Fix $N \in \mathbb{N}$. To begin, choose $r_1, t_1 \in S$ with $[t_1 r_1^{-1}] \neq \emptyset$ and set

$$x_1 = r_1 \cdot \lambda_{t_1} = \chi_{[t_1 r_1^{-1}]} \,.$$

Then $\rho(x_1) \in A_{r_1 S^\flat} \widehat{\otimes} E = \ell^1(r_1 S^\flat, E)$, and $1 = \|x_1\|_\infty = \|\pi \circ \rho(x_1)\|_\infty \leq \|\rho(x_1)\|_\pi$. Take a finite set $F_1 \subset r_1 S^\flat$ with $\|\chi_{F_1}\rho(x_1) - \rho(x_1)\|_\pi < 1/3$.

Now suppose that $x_1, \ldots, x_k$ and $\{F_1, \ldots, F_k\}$ are already constructed. Choose $r_{k+1} \in S$ with $r_{k+1} S^\flat \cap \bigcup_{i=1}^k F_i = \emptyset$. Set

$$G = \bigcup_{i=1}^{k} [t_i r_i^{-1}] \quad \text{and} \quad H = \bigcup_{s \in G} [(s r_{k+1}) r_{k+1}^{-1}] \,.$$

The set H is finite, so we can choose an element u in the complement. Set

$$t_{k+1} = u r_{k+1} \quad \text{and} \quad x_{k+1} = r_{k+1} \cdot \lambda_{t_{k+1}} = \chi_{[t_{k+1} r_{k+1}^{-1}]} \,.$$

Since $u \in [t_{k+1} r_{k+1}^{-1}]$ we have $\|\rho(x_{k+1})\|_\pi \geq 1$.

We shall show that the set $[t_{k+1} r_{k+1}^{-1}]$ is disjoint from G. Assume towards a contradiction that there exists $v \in [t_{k+1} r_{k+1}^{-1}] \cap G$. Then $v r_{k+1} = t_{k+1} = u r_{k+1}$, and so $u \in [(v r_{k+1}) r_{k+1}^{-1}] \subset H$. This is a contradiction, and therefore $[t_{k+1} r_{k+1}^{-1}] \cap G = \emptyset$, whence $\left\| \sum_{i=1}^{k+1} x_i \right\|_\infty = 1$. We have $\rho(x_{k+1}) \in A_{r_{k+1} S^\flat} \widehat{\otimes} E$. Take a finite set $F_{k+1} \subset r_{k+1} S^\flat$ with $\left\| \chi_{F_{k+1}} \rho(x_{k+1}) - \rho(x_{k+1}) \right\|_\pi < 1/3^{k+1}$.

In the final stage we may take $F_N = S \setminus \bigcup_{i=1}^{N-1} F_i$, so that the sets (F_i) form a partition of S. □

Theorem 6.2. *Let S be an infinite, weakly right cancellative semigroup. Suppose that $c_0(S)$ is projective in $\ell^1(S)$-**mod**. Then there exists a finite set $F \subset S$ such that, for each $r \in S$,*

$$rS^\flat \cap F \neq \emptyset \,.$$

Proof. We set $A_{S^\flat} = \ell^1(S^\flat)$ and $E = c_0(S)$.

Since E is projective in $\ell^1(S)$-**mod**, there exists a left $\ell^1(S)$-module morphism $\rho : E \to A_{S^\flat} \widehat{\otimes} E$ with $\pi \circ \rho = I_E$. Assume towards a contradiction that the condition is not satisfied, so that we can apply Lemma 6.1. Fix $N \in \mathbb{N}$, and let $x_1, \ldots, x_N$ and $\{F_1, \ldots, F_N\}$ be the elements corresponding to ρ given by Lemma 6.1.

Firstly, for each $m \in \mathbb{N}_N$ we have

$$\left\| \chi_{F_m} \rho \left(\sum_{i \neq m} x_i \right) \right\|_\pi \leq \sum_{i \neq m} \| \chi_{F_m} \rho(x_i) \|_\pi \leq \sum_{i \neq m} \left\| \chi_{S \setminus F_i} \rho(x_i) \right\|_\pi \leq \sum_{i \neq m} \frac{1}{3^i}$$

and $\| \chi_{F_m} \rho(x_m) \|_\pi \geq 1 - \frac{1}{3^m}$. Hence

$$\left\| \chi_{F_m} \rho \left(\sum_{i=1}^N x_i \right) \right\|_\pi \geq 1 - \frac{1}{3^m} - \sum_{i \neq m} \frac{1}{3^i} \geq 1 - \left(\frac{1}{1 - 1/3} - 1 \right) = \frac{1}{2} ,$$

and so,

$$\|\rho\| \geq \left\| \rho \left(\sum_{i=1}^N x_i \right) \right\|_\pi = \left\| \chi_{F_1} \rho \left(\sum_{i=1}^N x_i \right) \right\|_\pi + \cdots + \left\| \chi_{F_N} \rho \left(\sum_{i=1}^N x_i \right) \right\|_\pi \geq \frac{N}{2} .$$

This holds for each $N \in \mathbb{N}$, the required contradiction. □

Corollary 6.3. *Let S be a weakly right cancellative semigroup. Suppose that $c_0(S)$ is projective in $\ell^1(S)$-***mod***. Then the set*

$$\bigcup_{t \in S} [tt^{-1}]$$

is finite. Hence the set $E(S)$ is also finite.

Proof. Let F be the finite set given by Theorem 6.2. Take $t \in S$ and $u \in [tt^{-1}]$. Then either $t \in F$ or $f = ts$ for some $s \in S$ and $f \in F$. In the latter case we have $uf = uts = ts = f$, and so $u \in [ff^{-1}]$. Hence

$$\bigcup_{t \in S} [tt^{-1}] \subset \bigcup_{f \in F} [ff^{-1}] ,$$

and so the set $\bigcup_{t \in S} [tt^{-1}]$ is finite.

For each $p \in E(S)$ we have $p \in [pp^{-1}]$, hence the set $E(S)$ is finite. □

6.2 Main result for weakly cancellative semigroups

We shall adapt the argument that works for groups [4, Theorem 3.1], to show that if S is a weakly cancellative semigroup and $c_0(S)$ is projective, then S must be finite. The following technical looking lemma is a key to adapting the group argument. It is trivial that every element t in an infinite group satisfies the conditions in the lemma.

Lemma 6.4. *Let S be an infinite, weakly right cancellative semigroup. Suppose that $c_0(S)$ is projective in $\ell^1(S)$-***mod***. Then there exists an element $t \in S$ such that, for every finite set $F \subset S$, there exists $r \in S \setminus F$ with $[tr^{-1}] \neq \emptyset$.*

Proof. Assume towards a contradiction that the conclusion is false. Then, for every $t \in S$, there exists a finite set $F(t) \subset S$ such that $[tr^{-1}] = \emptyset$ for all $r \in S \setminus F(t)$.

Let F be the finite set given by Theorem 6.2. Take $s \in S \setminus F$. Then $su = f$ for some $f \in F$ and $u \in S$. Since $s \in [fu^{-1}]$ it must be that $u \in F(f)$, and hence

$$S \subset \bigcup_{f \in F} \bigcup_{u \in F(f)} [fu^{-1}] \cup F .$$

But the set on the right-hand side is finite, and so S is finite. This is a contradiction. Therefore the conclusion holds. □

Theorem 6.5. *Let S be a weakly cancellative semigroup such that $c_0(S)$ is projective in $\ell^1(S)$-**mod**. Then S is finite.*

Proof. Let $\rho : c_0(S) \to \ell^1(S^\flat) \widehat{\otimes} c_0(S)$ be a left $\ell^1(S)$-module morphism with $\pi \circ \rho = I_{c_0(S)}$.

Assume towards a contradiction that S is infinite. Let $t \in S$ be the element specified in Lemma 6.4. Fix $N \in \mathbb{N}$, and take a finite set $F \subset S^\flat$ with $\|\chi_F \rho(\lambda_t) - \rho(\lambda_t)\|_\pi < 1/N$. We shall construct elements $r_1, \dots, r_N \in S$ with the following properties:

(i) the sets $[tr_1^{-1}], \dots, [tr_N^{-1}]$ are pairwise disjoint and non-empty, and

(ii) the sets $r_1F, \dots, r_NF$ are pairwise disjoint.

To begin choose any $r_1 \in S$ with $[tr_1^{-1}] \neq \emptyset$. Now suppose that $r_1, \dots, r_k$ are already constructed. Set

$$X(k) = \bigcup_{i=1}^{k} \bigcup_{f,g \in F} [(r_i f)g^{-1}], \quad Y(k) = \bigcup_{i=1}^{k} [tr_i^{-1}], \quad Z(k) = \bigcup_{u \in Y(k)} [u^{-1}t] \,.$$

Since the sets $X(k)$ and $Z(k)$ are finite, we can use Lemma 6.4 to choose an element $r_{k+1} \in S \setminus X(k) \cup Z(k)$ with $[tr_{k+1}^{-1}] \neq \emptyset$. We now show that clauses (i) and (ii) are satisfied.

Take $1 \leq i < j \leq N$. Assume that there exists $u \in [tr_i^{-1}] \cap [tr_j^{-1}]$. Then $ur_j = t$, and so $r_j \in [u^{-1}t]$ for $u \in [tr_i^{-1}] \subset Y(j-1)$. Hence $r_j \in Z(j-1)$, which is a contradiction, giving clause (i). Next assume that there exists $v \in r_iF \cap r_jF$ so that $r_if = r_jg$ for some $f, g \in F$. But then $r_j \in [(r_if)g^{-1}] \subset X(j-1)$, which is a contradiction, and so clause (ii) holds.

For each $i \in \mathbb{N}_N$, since $[tr_i^{-1}] \neq \emptyset$, we have $\|r_i \cdot \rho(\lambda_t)\|_\pi \geq 1$ and we have the norm estimate

$$\|r_i \cdot \chi_F \rho(\lambda_t)\|_\pi \geq \|r_i \cdot \rho(\lambda_t)\|_\pi - \|r_i \cdot (\chi_F \rho(\lambda_t) - \rho(\lambda_t))\|_\pi \geq 1 - 1/N.$$

Now, we have

$$\begin{aligned} \|\rho\| &\geq \left\| \rho\left(\sum_{i=1}^{N} r_i \cdot \lambda_t \right) \right\|_\pi \geq \left\| \sum_{i=1}^{N} r_i \cdot \chi_F \rho(\lambda_t) \right\|_\pi - \left\| \sum_{i=1}^{N} r_i \cdot (\rho(\lambda_t) - \chi_F \rho(\lambda_t)) \right\|_\pi \\ &= \sum_{i=1}^{N} \|r_i \cdot \chi_F \rho(\lambda_t)\|_\pi - \left\| \sum_{i=1}^{N} r_i \cdot (\rho(\lambda_t) - \chi_F \rho(\lambda_t)) \right\|_\pi \\ &\geq N(1 - 1/N) - N/N = N - 2 \,. \end{aligned}$$

This holds for each $N \in \mathbb{N}$, the required contradiction. Therefore S is finite. □

Let S be a finite inverse semigroup. Then $\ell^1(S)$ is contractible [5, Theorem 8], and so every $E \in \ell^1(S)$-**mod** is projective. Hence we have the following.

Theorem 6.6. *Let S be a weakly cancellative inverse semigroup. Then $c_0(S)$ is projective in $\ell^1(S)$-**mod** if and only if S is finite.* □

Acknowledgements. This work forms part of the authors PhD thesis at the University of Leeds, which was supported by an EPSRC grant. The author wishes to thank his supervisor Professor H. G. Dales for his advice.

References

[1] Y. Choi, *Hochschild homology and cohomology of $\ell^1(\mathbb{Z}^k)$*, Published online October 2008, by Quarterly J. Math. (Oxford) (2008).

[2] H. G. Dales, *Banach algebras and automatic continuity*, London Mathematical Society Monographs **24** (Clarendon Press, Oxford), 2000.

[3] H. G. Dales, A. T.-M. Lau, and D. Strauss, *Banach algebras on semigroups and on their compactifications*, Memoirs of American Math. Soc., to appear.

[4] H. G. Dales and M. E. Polyakov, *Homological properties of modules over group algebras*, Proc. London Math. Soc. **(2) 63** (2004), 390–426.

[5] J. Duncan and I. Namioka, *Amenability of inverse semigroups and their semigroup algebras*, Proc. Royal Soc. Edinburgh, Section A, **80** (1978), 309–321.

[6] Brian E. Forrest, Hun Hee Lee, and Ebrahim Sami, *Projectivity of modules over Fourier algebras*, preprint, see arXiv:0807.4361.

[7] M. Ghandehari, H. Hatami, and N. Spronk, *Amenability constants for semilattice algebras*, Semigroup Forum **79** (2009), 279–297.

[8] F. Gourdeau, A. Pourabbas, and M. C. White, *Simplicial cohomology of some semigroup algebras*, Canadian Math. Bull. **50** (2007), no. 1, 50–70.

[9] N. Grønbæk, *Amenability of weighted discrete convolution algebras on cancellative semigroups*, Proc. Royal Soc. Edinburgh, Section A **110** (1988), 351–360.

[10] A. Ya. Helemskii, *The homology of Banach and topological algebras*, Kluwer Academic Publishers, Dordrecht, 1986.

[11] ______, *Banach and locally convex algebras*, Clarendon Press, Oxford, 1993.

[12] B. E. Johnson, *Cohomology in Banach algebras*, Memoirs American Math. Soc. **127** (1972).

[13] A. L. T. Paterson, *Amenability*, Mathematical Surveys and Monographs, Volume. 29, American Math. Soc., Providence, Rhode Island, 1988.

[14] P. Ramsden, *Homological properties of semigroup algebras*, Thesis, University of Leeds, 2009.

[15] V. Runde, *Lectures on amenability*, Lecture Notes in Mathematics 1774, Springer Verlag, 2002.

[16] Yu. V. Selivanov, *Coretraction problems and homological properties of Banach algebras*, Topological homology, (ed. A. Ya. Helemskii), Nova Science, Huntington, New York (2000), 145–200.