Non-injectivity of the ℓ^1-convolution algebra on an infinite free semilattice

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Abstract

Let F_X be the free meet-semilattice over a set X, (i.e., F_X is the collection of all finite subsets of X equipped the union operation). In this paper we prove that if X is infinite, then $\ell^1(F_X)$ is not an injective right module over $\ell^1(F_X)$.

1 Introduction

Let X be a set, let P_X be the power set of X, and let F_X the set of all finite subsets of X. Then P_X is a semilattice with the multiplication $st = s \cup t \ (s, t \in P_X)$, and F_X is a subsemilattice of P_X called the *free meet-semilattice* over X. In this paper we shall prove that, if X is an infinite set, then $\ell^1(P_X)$ and $\ell^1(F_X)$ are not injective Banach right modules over the Banach algebras $\ell^1(P_X)$ and $\ell^1(F_X)$, respectively.

This paper is a sequel to [6], which investigated the injectivity of $\ell^1(S)$ as a Banach right module over $\ell^1(S)$ for a semigroup S. In [6, Theorem 5.3] it is shown that if $\ell^1(S)$ is an injective right module, then S must satisfy a certain combinatorial condition. This condition is used to show, for example, that $\ell^1((\mathbb{N}, \max))$ is not injective. The semigroup F_X is an obvious and important example that *does* satisfy this condition, see [6, Example 5.7]. Hence our result shows that the necessary condition of [6, Theorem 5.3] is not sufficient.

2 Preliminaries

We shall follow the notation of [6], in particular the projective tensor product is denoted by $\widehat{\otimes}$, and $\mathcal{B}(E, F)$ denotes the space of all bounded linear operators between Banach spaces E and F.

2.1 Banach modules

We recall some definitions and results from Banach homology, for full details see [4] and [7]. Throughout this section A is a Banach algebra. We denote by A-**mod**, and by **mod**-A the categories of Banach left A-modules, and of Banach right A-modules, respectively.

For any Banach space E, we consider $A \widehat{\otimes} E \in A$-**mod** and $\mathcal{B}(A, E) \in$ **mod**-A in the usual way. Let $E \in A$-**mod**. Then we define the *canonical morphism* $\pi : A \widehat{\otimes} E \to E$ by

$$\pi(a \otimes x) = a \cdot x \quad (a \in A, \, x \in E).$$

Let $E \in$ **mod**-A be *faithful* ($x = 0$ whenever $x \cdot A = \{0\}$). Then we define the *canonical embedding* $\Pi : E \to \mathcal{B}(A, E)$ by the formula

$$\Pi(x)(a) = x \cdot a \quad (a \in A, \, x \in E).$$

Proposition 2.1. *Let A be a Banach algebra, and let $C > 0$.*

(i) *Let $F \in$ **mod**-A be faithful. Then F is C*-injective *if and only if there exists a right A module morphism $\rho : \mathcal{B}(A, F) \to F$ with $\rho \circ \Pi = I_F$ and $\|\rho\| \le C$.*

(ii) *Let $E \in A$-**mod** be essential ($\overline{AE} = E$). Then E is C*-projective *if and only if there exists a left A module morphism $\rho : E \to A \widehat{\otimes} E$ with $\pi \circ \rho = I_E$ and $\|\rho\| \le C$.* □

We shall use the following basic fact: if $E \in A$-**mod** is C-projective, then the dual module $E' \in$ **mod**-A is C-injective.

Let A and B be Banach algebras, let $E \in$ **mod**-A, and let $F \in$ **mod**-B. We regard $E \widehat{\otimes} F$ as a Banach right $(A \widehat{\otimes} B)$-module with the multiplication given by

$$(x \otimes y) \cdot (a \otimes b) = (x \cdot a) \otimes (y \cdot b) \quad (x \in E, y \in F, \, a \in A, \, b \in B)\,.$$

Proposition 2.2. *Let A and B be unital Banach algebras, and let $E \in$ **mod**-A and $F \in$ **mod**-B be unital modules. Suppose that $E \widehat{\otimes} F$ is C-injective in **mod**-$(A \widehat{\otimes} B)$. Then E is C-injective in **mod**-A.*

Proof. Since E and F are unital, $E \widehat{\otimes} F$ is faithful in **mod**-$(A \widehat{\otimes} B)$. Hence there is a right $(A \widehat{\otimes} B)$-module morphism $\rho : \mathcal{B}(A \widehat{\otimes} B, E \widehat{\otimes} F) \to E \widehat{\otimes} F$ with $\rho \circ \Pi_{E \widehat{\otimes} F} = I_{E \widehat{\otimes} F}$. Take $x_0 \in F_{[1]}$ and $\mu \in F'_{[1]}$ with $\langle x_0, \mu \rangle = 1$. Consider the diagram

$$\begin{array}{ccc} \mathcal{B}(A \widehat{\otimes} B, E \widehat{\otimes} F) & \xrightarrow{\ \rho\ } & E \widehat{\otimes} F \\ {\scriptstyle T \mapsto \widetilde{T}} \uparrow & & \downarrow {\scriptstyle I_E \otimes \mu} \\ \mathcal{B}(A, E) & \xrightarrow{\ \widetilde{\rho}\ } & E \end{array}\,,$$

where $\widetilde{T} = T \otimes \Pi_F(x_0)$ $(T \in \mathcal{B}(A, E))$, and set

$$\widetilde{\rho}(T) = (I_E \otimes \mu) \circ \rho(\widetilde{T}) \quad (T \in \mathcal{B}(A, E))\,.$$

We have

$$\widetilde{T \cdot a} = \widetilde{T} \cdot (a \otimes e_B) \quad (a \in A, T \in \mathcal{B}(A, E))\,,$$

and since F is unital we have

$$(I_E \otimes \mu)(z \cdot (a \otimes e_B)) = (I_E \otimes \mu)(z) \cdot a \quad (a \in A, z \in E \widehat{\otimes} F)\,.$$

It follows that $\widetilde{\rho}$ is a right A-module morphism. We also have

$$\widetilde{\Pi_E(x)} = \Pi_E(x) \otimes \Pi_F(x_0) = \Pi_{E \widehat{\otimes} F}(x \otimes x_0) \quad (x \in E)\,,$$

from which it follows that $\widetilde{\rho} \circ \Pi_E = I_E$. Therefore E is injective in **mod**-A.

The statement about the associated constants is clear. □

2.2 Semigroups and semigroup algebras

Let S be a semigroup. We shall use the following notation introduced by Grønbæk in [3]. For $s, t \in S$ we define the sets

$$[st^{-1}] = \{u \in S : ut = s\} \quad \text{and} \quad [t^{-1}s] = \{u \in S : tu = s\}\,.$$

The semigroup S is *weakly left* (respectively, *right*) *cancellative* if $[t^{-1}s]$ (respectively, $[st^{-1}]$) is finite for each $s, t \in S$, and *weakly cancellative* if it is both weakly left cancellative and weakly right cancellative.

Let X be a set. The empty set is the identity of P_X, which we denote by e. For $s, t \in P_X$, we have

$$[st^{-1}] = [t^{-1}s] = \{u : ut = s\} = \begin{cases} \emptyset \text{ if } t \not\subset s \\ \{s \setminus u : u \subset t\} \text{ if } t \subset s \end{cases} . \tag{1}$$

For $t \subset s \in F_X$ we have $|[t^{-1}s]| = 2^{|t|}$, and hence F_X is weakly cancellative. For each $t \in P_X$, we have $[tt^{-1}] = \{u : u \subset t\}$.

The Banach algebra $\ell^1(S)$ and the dual module $\ell^\infty(S)$

The *semigroup algebra* $\ell^1(S)$ is the completion in the ℓ^1-norm of the algebra $\mathbb{C}S$. The *convolution product* $\star$ on $\ell^1(S)$ is uniquely defined by requiring that $\delta_s \star \delta_t = \delta_{st}$ $(s, t \in S)$. These Banach algebras have been studied by many authors. A recent exposition is the memoir [1].

Let S be a semigroup, and let $E \in \ell^1(S)$-**mod** or $E \in$ **mod**-$\ell^1(S)$. We shall use the following more compact notation for the module actions

$$t \cdot x = \delta_t \cdot x, \quad x \cdot t = x \cdot \delta_t \quad (x \in E,\, t \in S)\,.$$

The dual actions of $\ell^1(S)$ on $\ell^1(S)' = \ell^\infty(S)$ are given by

$$(t \cdot \lambda)(s) = \lambda(st), \quad (\lambda \cdot t)(s) = \lambda(ts) \quad (s, t \in S,\, \lambda \in \ell^\infty(S))\,.$$

For $s \in S$, the indicator function of the set $\{s\}$ will be denoted by δ_s when considered as an element of $\ell^1(S)$, and by λ_s when considered as an element of $\ell^\infty(S)$. This notation implies that the left module actions satisfy

$$t \cdot \delta_s = \delta_{ts} \quad \text{and} \quad t \cdot \lambda_s = \chi_{[st^{-1}]} \quad (t \in S)\,.$$

The subspace $c_0(S)$ of $\ell^\infty(S)$ is a left $\ell^1(S)$-submodule if and only if S is weakly right cancellative, [1, Theorem 4.6].

2.3 Two easy lemmas

We note two combinatorial lemmas that we shall need later. Both follow easily from the binomial theorem. We omit the proofs, which can be found in [5]. For $m, n \in \mathbb{N} \cup \{0\}$. The symbol $\binom{n}{m}$ denotes the number of m-subsets of an n-set. We set $\binom{n}{m} = 0$ if $m > n$.

Lemma 2.3. *Let X be a finite set.*

(i)

$$\sum_{s \subset X} (-1)^{|s|} = \begin{cases} 1 \ \textit{if } X = \emptyset \\ 0 \ \textit{if } X \neq \emptyset \end{cases} .$$

(ii) *Let $t \subset X$. Then*

$$\sum_{s \subset X} (-1)^{|s \cap t|} = \begin{cases} 2^{|X|} \ \textit{if } t = \emptyset \\ 0 \ \textit{if } t \neq \emptyset \end{cases} .$$

(iii) *Let $t_1, t_2 \subset X$. Then*

$$\sum_{s \subset X} (-1)^{|s \cap t_1| + |s \cap t_2|} = \begin{cases} 2^{|X|} \ \textit{if } t_1 = t_2 \\ 0 \ \textit{if } t_1 \neq t_2 \end{cases} .$$

□

Lemma 2.4. *Take $a, b, N \in \mathbb{N}$ with $b \leq a < N$. Then*

$$\sum_{k=0}^{N-1} \binom{N-a}{k-b} (-1)^k = \begin{cases} (-1)^{N-1} & \textit{if } a = b \\ 0 & \textit{if } b < a \end{cases} .$$

□

3 $\ell^\infty(P_N)$ is a $2^{N/2}$-projective $\ell^1(P_N)$-module

Let X be a set. Clearly P_X and F_X only depend on the cardinality of X. For $N \in \mathbb{N}$, we set $P_N = P_{\mathbb{N}_N}$. The Banach algebra $\ell^1(P_N)$ is contractible and has a unique diagonal d with $\|d\| = 5^N$ [2, Example 1.6]. It follows that $\ell^1(P_N)$ is 5^N-injective in **mod**-$\ell^1(P_N)$. Let C_N be the minimum C such that $\ell^1(P_N)$ is C-injective in **mod**-$\ell^1(P_N)$. We shall find the exact value of C_N and show that $C_N \to \infty$ as $N \to \infty$. From this we can deduce that $\ell^1(F_X)$ and $\ell^1(P_X)$ are not injective right modules when X is infinite.

Clearly $\ell^1(P_N)$ is C-injective in **mod**-$\ell^1(P_N)$ if and only if $\ell^\infty(P_N)$ is C-projective in $\ell^1(P_N)$-**mod**. We prefer to work with the latter formulation of the problem.

3.1 The form of an $\ell^1(P_N)$-module morphism $\rho : \ell^\infty(P_N) \to \ell^1(P_N) \widehat{\otimes} E$

Let S be a finite semigroup, let E be a Banach space, and let $\rho \in \mathcal{B}(\ell^\infty(S), \ell^1(S) \widehat{\otimes} E)$. Then ρ is a left $\ell^1(S)$-module morphism if and only if

$$r \cdot \rho(\lambda_t) = \sum_{u \in [tr^{-1}]} \rho(\lambda_u) \quad (r, t \in S) . \tag{2}$$

For each $t \in S$, we can write

$$\rho(\lambda_t) = \sum_{s \in S} \delta_s \otimes x_s^t , \tag{3}$$

where $x_s^t \in E \ (s \in S)$. Then Equation (2) holds if and only if

$$\sum_{u \in [r^{-1}s]} x_u^t = \sum_{u \in [tr^{-1}]} x_s^u \quad (r, s, t \in S) . \tag{4}$$

We denote the expressions on the left- and right-hand sides of equation (4) by $L(r, s, t)$ and $R(r, s, t)$, respectively.

Now we specialise to the case $S = P_N$ for $N \in \mathbb{N}$. From Equation (1), $L(r, s, t)$ and $R(r, s, t)$ have the form

$$L(r, s, t) = \sum_{u \subset t} x_{s \setminus u}^t \quad (r \subset s) \quad \text{and} \quad R(r, s, t) = \sum_{u \subset r} x_s^{t \setminus u} \quad (r \subset t) . \tag{5}$$

Theorem 3.1. *Let $N \in \mathbb{N}$, and let E be a Banach space. Let $\rho \in \mathcal{B}(\ell^\infty(P_N),\ \ell^1(P_N) \widehat{\otimes} E)$. Then ρ is a left $\ell^1(P_N)$-module morphism if and only if there exists a set $\{x_s : s \in P_N\} \subset E$ such that*

$$\rho(\lambda_t) = \sum_{s \in P_N} \delta_s \otimes (-1)^{|t|+|s|} x_{t \cap s} \quad (t \in P_N) . \tag{6}$$

Proof. First suppose that (6) defines a map $\rho \in \mathcal{B}(\ell^\infty(P_N), \ell^1(P_N) \widehat{\otimes} E)$. We shall show that equation (4) holds. In the notation of (3) we have

$$x_s^t = (-1)^{|t|+|s|} x_{t \cap s} . \tag{7}$$

For $r, s, t \in P_N$ set

$$F(r,s,t) = (-1)^{|t|+|s|} \sum_{u \subset r} (-1)^{|u|} x_{(t\cap s)\setminus u} \,.$$

Fix $r, s \in P_N$ with $[r^{-1}s] \neq \emptyset$. Then substituting (7) into (5) gives $L(r,s,t) = F(r,s,t)$ $(t \in P_N)$. Similarly, for $t, r \in P_N$ with $[tr^{-1}] \neq \emptyset$ we have $R(r,s,t) = F(r,s,t)$ $(s \in P_N)$.

Now fix $r, t, s \in P_N$. Clearly (4) holds in the case where $[r^{-1}s] \neq \emptyset$ and $[tr^{-1}] \neq \emptyset$ since $L(r,s,t) = R(r,s,t) = F(r,s,t)$. The case where $[r^{-1}s] = [tr^{-1}] = \emptyset$ is also clear since $L(r,s,t) = R(r,s,t) = 0$. Suppose that $[r^{-1}s] \neq \emptyset$ and $[tr^{-1}] = \emptyset$. The latter condition implies that $R(r,s,t) = 0$ and that $r \cap (\mathbb{N}_N \setminus t) \neq \emptyset$. Then we have

$$\begin{aligned} L(r,s,t) = F(r,s,t) &= (-1)^{|t|+|s|} \sum_{u \subset r\cap t\cap s} (-1)^{|u|} \left(\sum_{v \subset r \cap (\mathbb{N}_N \setminus t \cap s)} (-1)^{|v|} \right) x_{t\cap(s\setminus v)} \\ &= 0 \quad (\text{by Lemma 2.3(i)})\,, \end{aligned}$$

and so $L(r,s,t) = R(r,s,t)$ in this case. Exactly the same argument works in the case where $[r^{-1}s] = \emptyset$ and $[tr^{-1}] \neq \emptyset$. Therefore by (4) ρ is a left $\ell^1(P_N)$-module morphism.

Now we turn to the converse. Take $\rho : \mathcal{B}(\ell^\infty(P_N), \ell^1(P_N) \widehat{\otimes} E)$, and let $\{x_s^t : s, t \in S\} \subset E$ be the set of elements given in (3). We first prove that for each $t \in P_N$,

$$x_v^t = x_t^v \quad (v \subset t)\,. \tag{8}$$

The proof is by induction on $|t \setminus v| = |t| - |v|$. The result is clear if $|t \setminus v| = 0$ i.e., if $v = t$. Assume that $x_v^t = x_t^v$ for all $v \subset t$ with $|t \setminus v| \leq k$. Take $v \subset t$ with $|t \setminus v| = k+1$. Set $s = t$, $r = t \setminus v$ in (4) to obtain

$$\sum_{u \subset t\setminus v} x_{t\setminus u}^t = \sum_{u \subset t \setminus v} x_t^{t\setminus u}\,.$$

If $u \subsetneq t \setminus v$, then $|t \setminus (t \setminus u)| = |u| < |t \setminus v| = k+1$. Hence for $u \subsetneq t \setminus v$ we have $|t \setminus (t \setminus u)| \leq k$. By the induction hypothesis all such terms in the above sum cancel leaving $x_v^t = x_t^v$, and the result is proved.

Next we shall show that

$$x_v^t = (-1)^{|t|+|v|} x_{t\cap v}^{t\cap v} \quad (t, v \in P_N)\,. \tag{9}$$

The proof is by induction on $|t|$.

We first consider the case where $|t| = 0$ i.e., $t = e$. We have to show that $x_v^e = (-1)^{|v|} x_e^e$ for all $v \in P_N$. We shall prove this by induction on $|v|$. The result is clear if $|v| = 0$. Assume that $x_v^e = (-1)^{|v|} x_e^e$ for all v with $|v| \leq k$. Take $v \in P_N$ with $|v| = k+1$. Set $r = s = v$ in (4). Then $R(v, v, e) = 0$ and we have

$$\begin{aligned} x_v^e = -\sum_{u \subsetneq v} x_u^e = -\sum_{k=0}^{|v|-1} \sum_{\{u\subset v, |u|=k\}} x_u^e &= -\sum_{k=0}^{|v|-1} \binom{|v|}{k} (-1)^k x_e^e \\ = -(0 - (-1)^{|v|}) x_e^e = (-1)^{|v|} x_e^e\,, \end{aligned}$$

which proves that (9) holds in the case where $t = e$.

Now assume that $x_v^t = (-1)^{|t|+|v|} x_{t\cap v}^{t\cap v}$ for all $t \in P_N$ with $|t| \leq k$ and all $v \in P_N$. Take $t \in P_N$ with $|t| = k+1$. We have to show that $x_v^t = (-1)^{|t|+|v|} x_{t\cap v}^{t\cap v}$ for all $v \in P_N$. We shall use induction on $|v|$. By (8) the result holds if $|v| = 0$. Indeed, for any $v \subset t$ by (8) and the induction hypothesis on t, we have

$$x_v^t = x_t^v = (-1)^{|v|+|t|} x_v^v\,,$$

and so the result holds in this case. Now assume that the result holds for all v with $|v| \leq k$. Take $v \in P_N$ with $|v| = k+1$, and set $t_0 = v \cap t$. Set $r = s = v$ in (4). We may suppose that $v \not\subset t$, so that $R(v,v,t) = 0$. Then we have

$$\begin{aligned} x_v^t &= -\sum_{u \subsetneq v} x_u^t = -\sum_{k=0}^{|v|-1} \sum_{\{u\subset v, |u|=k\}} x_u^t = -\sum_{k=0}^{|v|-1} \sum_{t_1 \subset t_0} \sum_{\{u \subset v, |u|=k, u\cap t = t_1\}} x_u^t \\ &= -\sum_{k=0}^{|v|-1} \sum_{t_1\subset t_0} \binom{|v|-|t_0|}{k-|t_1|} (-1)^{|t|+k} x_{t_1}^{t_1} = -(-1)^{|t|+|v|-1} x_{t_0}^{t_0} = (-1)^{|t|+|v|} x_{t_0}^{t_0}\,, \end{aligned}$$

where, in the last line, we have used Lemma 2.4. Therefore (9) holds for all $t, v \in P_N$.

Finally, we set $x_v = x_v^v$ $(v \in P_N)$, so that, for each $t \in P_N$, we have

$$\rho(\lambda_t) = \sum_{s\in P_N} \delta_s \otimes x_s^t = \sum_{s \in P_N} \delta_s \otimes (-1)^{|s|+|t|} x_{s\cap t}^{s \cap t} = \sum_{s\in P_N} \delta_s \otimes (-1)^{|s|+|t|} x_{s\cap t}\,,$$

as required. □

3.2 The identity $\pi \circ \rho = I_{\ell^\infty(P_N)}$

Let $N \in \mathbb{N}$, and set $E = \ell^\infty(P_N)$. Let ρ be as in (6). We shall impose the identity $\pi \circ \rho = I_E$ and see what restrictions this puts on the set of elements $\{x_s : s \in S\}$ given by (6). The identity $\pi\circ\rho = I_E$ holds if and only if

$$\pi \circ \rho(\lambda_t) = \lambda_t \quad (t \in P_N)\,. \tag{10}$$

For each $s \in P_N$ we can write

$$x_s = \sum_{r \in P_N} \beta_r^s \lambda_r\,, \tag{11}$$

where $\{\beta_r^s : r \in P_N\} \subset \mathbb{C}$. For each $t \in P_N$, equation (10) becomes

$$\begin{aligned} \lambda_t &= \sum_{s\in P_N} s \cdot x_s^t = \sum_{s \in P_N} (-1)^{|t|+|s|} s \cdot x_{t\cap s} = \sum_{s\in S} (-1)^{|t|+|s|} \sum_{r \in P_N} \beta_r^{t\cap s} \chi_{[rs^{-1}]} \\ &= \sum_{u \in P_N} \left(\sum_{\{(r,s): u \in [rs^{-1}]\}} (-1)^{|t|+|s|} \beta_r^{t\cap s} \right) \lambda_u\,. \end{aligned}$$

This holds if and only if

$$\begin{aligned} \delta_{t,u} &= \sum_{\{(r,s):u\in[rs^{-1}]\}} (-1)^{|t|+|s|} \beta_r^{t\cap s} = \sum_{r \in uS} \sum_{s \in [u^{-1}r]} (-1)^{|t|+|s|} \beta_r^{t\cap s} \\ &= (-1)^{|t|} \sum_{t \supset u} \sum_{s \subset u} (-1)^{|r\setminus s|} \beta_r^{t\cap(r\setminus s)} \quad (u \in P_N)\,, \end{aligned}$$

where $\delta_{t,u} = 1$ if $t = u$ and $\delta_{t,u} = 0$ if $t \neq u$. This can be rewritten as

$$(-1)^{|t|}\delta_{t,u} = \sum_{r \supset u} (-1)^{|r|} \sum_{s \subset u} (-1)^{|s|} \beta_r^{t\cap(r\setminus s)} \quad (u \in P_N)\,. \tag{12}$$

Surprisingly, this equation automatically holds for certain $t, u \in P_N$.

Lemma 3.2. *Equation* (12) *holds for all* $t, u \in P_N$ *with* $u \setminus t \neq \emptyset$.

Proof. For $r \supset u$ the inner sum on the right hand side of (12) is

$$\sum_{s \subset u} (-1)^{|s|} \beta_r^{t \cap (r \setminus s)} = \sum_{s_1 \subset u \cap t} \sum_{s_2 \subset u \setminus t} (-1)^{|s_1| + |s_2|} \beta_r^{t \cap (r \setminus s_1)}$$
$$= \begin{cases} 0 \text{ if } u \setminus t \neq \emptyset \\ \sum_{s_1 \subset u \cap t} (-1)^{|s_1|} \beta_r^{t \cap (r \setminus s_1)} \text{ if } u \setminus t = \emptyset \end{cases} \quad \text{(by Lemma 2.3(i))}.$$

Hence both sides of (12) are equal to 0. $\square$

Corollary 3.3. *The identity* $\pi \circ \rho = I_E$ *holds if and only if equation* (12) *holds for all* $t, u \in P_N$ *with* $u \subset t$. $\square$

3.3 Main result

We are ready to prove the main result of this section. The last ingredient we need is an expression for $\|\rho(x)\|$. It is convenient to consider

$$x = \sum_{t \in P_N} (-1)^{|t|} \gamma_t \lambda_t \in \ell^\infty(P_N),$$

where $\{\gamma_t : t \in P_N\} \subset \mathbb{C}$. We have

$$\rho(x) = \sum_{s \in P_N} \delta_s \otimes (-1)^{|s|} \sum_{t \in P_N} \gamma_t x_{t \cap s}.$$

By the identification $\ell^1(P_N) \widehat{\otimes} \ell^\infty(P_N) = \ell^1(P_N, \ell^\infty(P_N))$, we have

$$\|\rho(x)\| = \sum_{s \in P_N} \left\| \sum_{t \in P_N} \gamma_t x_{t \cap s} \right\|. \tag{13}$$

Theorem 3.4. *Let* $N \in \mathbb{N}$. *Then* $\ell^\infty(P_N)$ *is* $2^{N/2}$*-projective in* $\ell^1(P_N)$-**mod**.

Proof. We define a map $\rho : \ell^\infty(P_N) \to \ell^1(P_N) \widehat{\otimes} \ell^\infty(P_N)$ via (6) and (11) by setting

$$\beta_r^s = \frac{1}{|P_N|} (-1)^{|r|+|s|} \quad (r, s \in P_N).$$

By Theorem 3.1, ρ is a left $\ell^1(P_N)$-module morphism. We check that (12) holds. Take $t, u \in P_N$ with $u \subset t$. We shall denote the right-hand side of (12) by $\Sigma(t, u)$. We have

$$|P_N| \, \Sigma(t, u) = \sum_{r \supset u} \sum_{s \subset u} (-1)^{|s| + |t \cap (r \setminus s)|}.$$

For each r and s in the sum above, we have $s \subset r \cap t$, and so $|t \cap (r \setminus s)| = |t \cap r| - |s|$. This gives

$$|P_N| \, \Sigma(t, u) = \sum_{r \supset u} \sum_{s \subset u} (-1)^{|t \cap r|} = \sum_{r \supset u} 2^{|u|} (-1)^{|t \cap r|} = 2^{|u|} \sum_{r \subset \mathbb{N}_N \setminus u} (-1)^{|u| + |(t \setminus u) \cap r|}$$
$$= 2^{|u|} (-1)^{|u|} 2^{N - |u|} \delta_{\emptyset (t \setminus u)} \quad \text{(by Lemma 2.3(ii))}$$
$$= |P_N| \, (-1)^{|t|} \delta_{tu}.$$

Therefore (12) holds. By Corollary 3.3, we have $\pi \circ \rho = I_{\ell^\infty(P_N)}$.

Next we estimate $\|\rho\|$. For $x = \sum_{t \in P_N} (-1)^{|t|} \gamma_t \lambda_t \in \ell^\infty(P_N)$ and $s \in P_N$ we first calculate

$$\left\| \sum_{t \in P_N} \gamma_t x_{t \cap s} \right\| = \left\| \sum_{r \in P_N} \left(\sum_{t \in P_N} \gamma_t \beta_r^{t \cap s} \right) \lambda_r \right\| = \frac{1}{|P_N|} \left\| \sum_{r \in P_N} \left(\sum_{t \in P_N} \gamma_t (-1)^{|t \cap s| + |r|} \right) \lambda_r \right\|$$
$$= \frac{1}{|P_N|} \left| \sum_{t \in P_N} \gamma_t (-1)^{|t \cap s|} \right| .$$

By (13) we have

$$\begin{aligned}
|P_N| \, \|\rho(x)\| &= \sum_{s \in P_N} \left| \sum_{t \in P_N} \gamma_t (-1)^{|t \cap s|} \right| \\
&\leq |P_N|^{1/2} \left(\sum_{s \in P_N} \left| \sum_{t \in P_N} \gamma_t (-1)^{|t \cap s|} \right|^2 \right)^{1/2} \qquad \text{(by Hölder's inequality)} \\
&= |P_N|^{1/2} \left[\sum_{s \in P_N} \left(\sum_{t \in P_N} \gamma_t (-1)^{|t \cap s|} \right) \left(\sum_{t \in P_N} \overline{\gamma}_t (-1)^{|t \cap s|} \right) \right]^{1/2} \\
&= |P_N|^{1/2} \left[\sum_{t, r \in P_N} \left(\sum_{s \in P_N} (-1)^{|t \cap s| + |r \cap s|} \right) \gamma_t \overline{\gamma_r} \right]^{1/2} \\
&= |P_N|^{1/2} \left[|P_N| \sum_{t \in P_N} |\gamma_t|^2 \right]^{1/2} \qquad \text{(by Lemma 2.3(iii))} \\
&= |P_N| \, \|x\|_2 \leq |P_N|^{3/2} \, \|x\|_\infty .
\end{aligned}$$

Therefore $\|\rho\| \leq |P_N|^{1/2} = 2^{N/2}$, and the result is proved. □

4 $2^{N/2}$ is the best possible constant of projectivity

Let $N \in \mathbb{N}$. In this section we shall prove that $\ell^\infty(P_{2N})$ is not C-projective in $\ell^1(P_{2N})$-**mod** for any $1 < C < 2^N$.

We now fix throughout this section $N \in \mathbb{N}$ and a left $\ell^1(P_N)$-module morphism

$$\rho : \ell^\infty(P_{2N}) \to \ell^1(P_{2N}) \widehat{\otimes} \ell^\infty(P_{2N}) \quad \text{with} \quad \pi \circ \rho = I_{\ell^\infty(P_{2N})} . \tag{14}$$

We shall prove that $\|\rho\| \geq 2^N$. Let $\{x_t : t \in P_{2N}\} \subset \ell^\infty(P_{2N})$ be the set of elements corresponding to ρ given in Theorem 3.1.

We set

$$p_n = \{2n-1, 2n\} \in P_{2N} \quad (n \in \mathbb{N}_N) .$$

We inductively define sets of complex numbers $\{\{\gamma_t^{2n} : t \in P_{2n}\} : n \in \mathbb{N}_N\}$ by setting

$$\gamma_e^2 = \gamma_1^2 = \gamma_2^2 = 1, \quad \gamma_{p_1}^2 = -1 ,$$

and for each $2 \leq n \leq N$ setting

$$\gamma_t^{2n} = \begin{cases} \gamma_{t \setminus p_n}^{2(n-1)} & \text{if } p_n \not\subset t \\ -\gamma_{t \setminus p_n}^{2(n-1)} & \text{if } p_n \subset t \end{cases} \qquad (t \in P_{2n}) .$$

Now we define a vector $y \in \ell^\infty(P_{2N})_{[1]}$ by

$$y = \sum_{t \in P_{2N}} \gamma_t^{2N} (-1)^{|t|} \lambda_t \,. \tag{15}$$

It is convenient to introduce the following notation. For $n \in \mathbb{N}_N$ and $s \in P_{2n}$, we set

$$\Lambda(s; 2n) = \sum_{t \in P_{2n}} \gamma_t^{2n} x_{t \cap s} \,.$$

Recall from (13) that

$$\|\rho(y)\| = \sum_{s \in P_{2N}} \|\Lambda(s; 2N)\| \,. \tag{16}$$

We shall prove that $\|\rho(y)\| \geq 2^N$. The proof consists of the following steps:

(i) First we derive a recurrence relation between the $\Lambda(s; 2n)$'s.

(ii) Then we prove a formula for $\Lambda(s; 2N)$.

(iii) Next we estimate 'part' of (16).

(iv) Finally we add up the 'parts' to get an estimate for (16).

4.1 A recurrence relation

Consider the vector space $\mathrm{lin}\,\{(s, x_s) : s \in P_{2N}\}$ of formal linear combinations. We define an action $*$ of $\ell^1(P_{2N})$ on this space by setting

$$(s, x_s) * \delta_t = (s \cup t, x_{s \cup t}) \quad (s, t \in P_{2N}) \,.$$

In order to simplify our notation we shall identify x_s with (s, x_s), and write $x_s * \delta_t = x_{s \cup t}$. For example, with this notation we have

$$\Lambda(s; 2n) * (\alpha \delta_{t_1} + \beta \delta_{t_2}) = \alpha \sum_{t \in P_{2n}} \gamma_t^{2n} x_{(t \cap s) \cup t_1} + \beta \sum_{t \in P_{2n}} \gamma_t^{2n} x_{(t \cap s) \cup t_2} \,.$$

Lemma 4.1. *Let $2 \leq n \leq N$, and let $s \in P_{2n}$. If $p_n \not\subset s$, then*

$$\Lambda(s; 2n) = 2\Lambda(s \setminus p_n; 2(n-1)) \,. \tag{17}$$

If $p_n \subset s$, then

$$\Lambda(s; 2n) = \Lambda(s \setminus p_n; 2(n-1)) * (\delta_e + \delta_{\{2n-1\}} + \delta_{\{2n\}} - \delta_{p_n}) \,. \tag{18}$$

Proof. We have

$$\begin{aligned}
\Lambda(s; 2n) &= \sum_{t \in P_{2(n-1)}} \gamma_{t \cup p_n}^{2n} x_{(t \cup p_n) \cap s} + \sum_{t \in P_{2(n-1)}} \gamma_{t \cup \{2n-1\}}^{2n} x_{(t \cup \{2n-1\}) \cap s} \\
&\quad + \sum_{t \in P_{2(n-1)}} \gamma_{t \cup \{2n\}}^{2n} x_{(t \cup \{2n\}) \cap s} + \sum_{t \in P_{2(n-1)}} \gamma_t^{2n} x_{t \cap s} \\
&= - \sum_{t \in P_{2(n-1)}} \gamma_t^{2(n-1)} x_{(t \cup p_n) \cap s} + \sum_{t \in P_{2(n-1)}} \gamma_t^{2(n-1)} x_{(t \cup \{2n-1\}) \cap s} \\
&\quad + \sum_{t \in P_{2(n-1)}} \gamma_t^{2(n-1)} x_{(t \cup \{2n\}) \cap s} + \sum_{t \in P_{2(n-1)}} \gamma_t^{2n} x_{t \cap s} \,.
\end{aligned}$$

If $p_n \subset s$, then we have the result stated. If $s \cap p_n = \{2n\}, \{2n-1\}$ or e, then the first term cancels with the second, third or fourth term respectively, giving the result in this case. ☐

4.2 A formula for $\Lambda(s;2N)$

For $s \in P_{2N}$ we set

$$F(s) = \sum_{t \subset s} (-1)^{|t|} \delta_t \in \ell^1(P_{2N}) \quad \text{and} \quad \Gamma(s) = \sum_{t \subset s} (-1)^{|t|} x_t \in \ell^\infty(P_{2N}).$$

Lemma 4.2. *Let $s, t \in P_{2N}$ with $s \cap t = \emptyset$. Then*

$$\Gamma(s) * F(t) = \Gamma(s \cup t).$$

Proof. We calculate

$$\Gamma(s \cup t) = \sum_{r \subset s \cup t} (-1)^{|r|} x_r = \sum_{t_0 \subset t} \sum_{s_0 \subset s} (-1)^{|t_0|+|s_0|} x_{t_0 \cup s_0} = \sum_{t_0 \subset t} (-1)^{|t_0|} \Gamma(s) * \delta_{t_0}$$

$$= \Gamma(s) * \left(\sum_{t_0 \subset t} (-1)^{|t_0|} \delta_{t_0} \right) = \Gamma(s) * F(t). \qquad \square$$

For $s \in P_{2N}$ we set

$$Y(s) = \{ n \in \mathbb{N}_N : p_n \subset s \},$$

and, for each $0 \le k \le N$, we let $Y_k(s)$ denote the family of k-subsets of $Y(s)$.

Lemma 4.3. *Let $s \in P_{2N}$. Then*

$$\Lambda(s;2N) = \sum_{k=1}^{N} 2^{N-k} (-1)^k \sum_{\{n_1,\dots,n_k\} \in Y_k(s)} \Gamma(p_{n_1} \cup \cdots \cup p_{n_k}) + 2^N x_e. \tag{19}$$

Proof. For $n \in \mathbb{N}_N$, let $Q(n)$ denote the statement that for all $s \in P_{2n}$, the following equation holds:

$$\Lambda(s;2n) = \sum_{k=1}^{n} 2^{n-k} (-1)^k \sum_{\{n_1,\dots,n_k\} \in Y_k(s)} \Gamma(p_{n_1} \cup \cdots \cup p_{n_k}) + 2^n x_e. \tag{20}$$

We shall prove by induction on n that $Q(n)$ is true for each $n \in \mathbb{N}_N$.

First consider the case where $n = 1$, so that $P_{2n} = \{e, \{1\}, \{2\}, p_1\}$. If $s = e, \{1\}$ or $\{2\}$, then it is easily checked that

$$\Lambda(s;2) = 2x_e,$$

which agrees with (20) since $Y_1(s) = \emptyset$ for $s = e, \{1\}$, or $\{2\}$. A similar direct check shows that

$$\Lambda(p_1;2) = \gamma_e^2 x_e + \gamma_{\{1\}}^2 x_{\{1\}} + \gamma_{\{2\}}^2 x_{\{2\}} + \gamma_{\{p_1\}}^2 x_{\{p_1\}} = -\Gamma(p_1) + 2x_e,$$

and so $Q(1)$ is true.

Now assume that $Q(n)$ holds for a fixed $n \in \mathbb{N}_{N-1}$. Take $s \in P_{2(n+1)}$. First suppose that $p_{n+1} \not\subset s$. Then by (17) we have

$$\begin{aligned}
\Lambda(s;2(n+1)) &= 2\Lambda(s \setminus p_{n+1};2n) \\
&= 2 \left(\sum_{k=1}^{n} 2^{n-k} (-1)^k \sum_{\{n_1,\dots,n_k\} \in Y_k(s \setminus p_{n+1})} \Gamma(p_{n_1} \cup \cdots \cup p_{n_k}) + 2^n x_e \right) \\
&= \sum_{k=1}^{n+1} 2^{n+1-k} (-1)^k \sum_{\{n_1,\dots,n_k\} \in Y_k(s)} \Gamma(p_{n_1} \cup \cdots \cup p_{n_k}) + 2^{n+1} x_e.
\end{aligned}$$

The last line follows since $Y_k(s) = Y_k(s \setminus p_{n+1})$ $(k \in \mathbb{N}_n)$ and $Y_{n+1}(s) = \emptyset$.

Now suppose that $p_{n+1} \subset s$. By (18) and Lemma 4.2 we have

$$\begin{aligned}
&\Lambda(s; 2(n+1)) = 2\Lambda(s \setminus p_{n+1}; 2n) * (2\delta_e - F(p_{n+1})) \\
&= \left(\sum_{k=1}^{n} 2^{n-k}(-1)^k \sum_{\{n_1,\dots,n_k\} \in Y_k(s \setminus p_{n+1})} \Gamma(p_{n_1} \cup \cdots \cup p_{n_k}) + 2^n x_e \right) * (2\delta_e - F(p_{n+1})) \\
&= \sum_{k=1}^{n} 2^{n+1-k}(-1)^k \sum_{\{n_1,\dots,n_k\} \in Y_k(s \setminus p_{n+1})} \Gamma(p_{n_1} \cup \cdots \cup p_{n_k}) + 2^{n+1} x_e \\
&\quad + \sum_{k=1}^{n} 2^{n+1-(k+1)}(-1)^{k+1} \sum_{\{n_1,\dots,n_k\} \in Y_k(s \setminus p_{n+1})} \Gamma(p_{n_1} \cup \cdots \cup p_{n_k} \cup p_{n+1}) - 2^n \Gamma(p_{n+1})\,.
\end{aligned}$$

This last line is a partition of the sum in (20). Therefore $Q(n+1)$ is true.

Therefore by induction $Q(n)$ is true for all $n \in \mathbb{N}_N$. In particular $Q(N)$ is true. □

4.3 Norm estimates

Lemma 4.4. *Let $r \in P_{2N}$. Then*

$$\|\Gamma(r)\| \geq \frac{1}{2^{2N-|r|}}\,.$$

Proof. We define $x = \sum_{t \in P_{2N}} (-1)^{|t|} \gamma_t \lambda_t \in \ell^\infty(P_{2N})$ by

$$\gamma_t = \begin{cases} (-1)^{|t|} & \text{if } t \subset r \\ 0 & \text{otherwise} \end{cases} \qquad (t \in P_{2N})\,.$$

First observe that for $s \in P_{2N}$ we have

$$\begin{aligned}
\sum_{t \subset r} \gamma_t x_{t \cap s} &= \sum_{s_1 \subset s \cap r} \left(\sum_{\{t \subset r : t \cap s = s_1\}} (-1)^{|t|} \right) x_{s_1} = \sum_{s_1 \subset s \cap r} \left(\sum_{t \subset r \setminus s} (-1)^{|s_1| + |t|} \right) x_{s_1} \\
&= \begin{cases} \sum_{s_1 \subset r} (-1)^{|s_1|} x_{s_1} & \text{if } r \subset s \\ 0 & \text{if } r \not\subset s \end{cases} \qquad \text{(by Lemma 2.3(i))}\,.
\end{aligned}$$

Since $1 = \|x\| = \|\pi \circ \rho(x)\| \leq \|\rho(x)\|$, by (13) we have

$$1 \leq \sum_{s \in P_{2N}} \left\| \sum_{t \subset r} \gamma_t x_{t \cap s} \right\| = \sum_{s \supset r} \left\| \sum_{s_1 \subset r} (-1)^{|s_1|} x_{s_1} \right\| = 2^{2N-|r|} \left\| \sum_{t \subset r} (-1)^{|t|} x_t \right\|,$$

which gives the result. □

Lemma 4.5. *Let $s \in P_{2N}$. Then*

$$\sum_{\{n_1,\dots,n_k\} \subset Y(s)} \|\Lambda(s \setminus p_{n_1} \cup \cdots \cup p_{n_k}; 2N)\| \geq \frac{1}{2^{N-|Y(s)|}}\,. \tag{21}$$

Proof. Set $Y = Y(s)$ and $\Lambda(t) = \Lambda(t; 2N)$ $(t \in P_{2N})$. We define

$$\sum_{\{n_1,\ldots,n_0\}\in Y_0(s)} \Gamma(p_{n_1} \cup \cdots \cup p_{n_0}) = x_e$$

so that (19) can be combined into a single sum.

By (19) we have

$$\sum_{\{n_1,\ldots,n_k\}\subset Y} \|\Lambda(s \setminus p_{n_1} \cup \cdots \cup p_{n_k})\| = \|\Lambda(s)\| + \sum_{\substack{\{n_1,\ldots,n_k\}\subset Y\\ \{n_1,\ldots,n_k\}\neq\emptyset}} \|\Lambda(s \setminus p_{n_1} \cup \cdots \cup p_{n_k})\|$$

$$\geq \left\|2^{N-|Y|}\Gamma\left(\cup_{n\in Y} p_n\right)\right\| - \left\|\sum_{k=0}^{|Y|-1} 2^{N-k}(-1)^k \sum_{\{n_1,\ldots,n_k\}\subset Y_k} \Gamma(p_{n_1} \cup \cdots \cup p_{n_k})\right\|$$

$$+ \sum_{\substack{\{n_1,\ldots,n_k\}\subset Y\\ \{n_1,\ldots,n_k\}\neq\emptyset}} \|\Lambda(s \setminus p_{n_1} \cup \cdots \cup p_{n_k})\| \quad \text{by (19)}\,.$$

By Lemma 4.4 we have

$$\left\|2^{N-|Y|}\Gamma(\cup_{n\in Y} p_n)\right\| \geq \frac{2^{N-|Y|}}{2^{2N-2|Y|}} = \frac{1}{2^{N-|Y|}}\,.$$

Hence the result will follow if we can show that

$$\left\|\sum_{k=0}^{|Y|-1} 2^{N-k}(-1)^k \sum_{\{n_1,\ldots,n_k\}\subset Y_k} \Gamma(p_{n_1} \cup \cdots \cup p_{n_k})\right\| \leq \sum_{\substack{\{n_1,\ldots,n_k\}\subset Y\\ \{n_1,\ldots,n_k\}\neq\emptyset}} \|\Lambda(s \setminus p_{n_1} \cup \cdots \cup p_{n_k})\|\,.$$

This inequality follows from the identity

$$\sum_{k=0}^{|Y|-1} 2^{N-k}(-1)^k \sum_{\{n_1,\ldots,n_k\}\subset Y_k} \Gamma(p_{n_1} \cup \cdots \cup p_{n_k}) = \sum_{l=1}^{|Y|}(-1)^l \sum_{\{n_1,\ldots,n_l\}\in Y_l} \Lambda(s \setminus p_{n_1} \cup \cdots \cup p_{n_l})\,.$$

To see that this is true, take $\{n_1, \ldots, n_k\} \subset Y$. The coefficient of $\Gamma(p_{n_1} \cup \cdots \cup p_{n_k})$ on the left-hand side is $2^{N-k}(-1)^k$, and on the right-hand side, using (19), it is

$$\sum_{l=1}^{|Y|-k} (-1)^l \binom{|Y|-k}{l} 2^{N-k}(-1)^k = 2^{N-k}(-1)^k\,,$$

which gives the result. □

4.4 Main results

Theorem 4.6. *Let $M \in \mathbb{N}$. Then $\ell^\infty(P_M)$ is $2^{M/2}$-projective in $\ell^1(P_M)$-**mod**, but not C-projective for any $1 < C < 2^{\lfloor M/2\rfloor}$.*

Proof. By Theorem 3.4, $\ell^\infty(P_M)$ is $2^{M/2}$-projective in $\ell^1(P_M)$-**mod**.

Suppose first that $M = 2N$ for some $N \in \mathbb{N}$, and that $\ell^\infty(P_{2N})$ is C-projective in $\ell^1(P_{2N})$-**mod**. Let ρ be as in (14), and let y be as in (15).

We call a set $s \in P_{2N}$ *special* if, for each $n \in \mathbb{N}_N$, either $2n-1 \in s$ or $2n \in s$. Let $0 \le k \le N$. Then a set $s \in P_{2N}$ is a *special set of order* k if s is a special set and $|Y(s)| = k$. The collection of special sets of order k is denoted by $\mathcal{S}_k$. We have $|\mathcal{S}_k| = \binom{N}{k} 2^{N-k}$. The important fact about special sets is that for every $t \in P_{2N}$ there exists a unique special set s with $t = s \setminus p_{n_1} \cup \cdots \cup p_{n_l}$ for some $\{n_1, \dots, n_l\} \subset Y(s)$. Combining these facts with Lemma 4.5 gives

$$\begin{aligned} \|\rho\| \ge \|\rho(y)\| &= \sum_{s \in P_{2N}} \|\Lambda(s; 2N)\| \quad \text{by (16)} \\ &= \sum_{k=0}^{N} \sum_{s \in \mathcal{S}_k} \sum_{\{n_1,\dots,n_l\} \subset Y(s)} \|\Lambda(s \setminus p_{n_1} \cup \cdots \cup p_{n_l}; 2N)\| \\ &\ge \sum_{k=0}^{N} \binom{N}{k} 2^{N-k} \frac{1}{2^{N-k}} = 2^N . \end{aligned}$$

Therefore $C \ge 2^N$, and the result is proved in the case where M is even.

Now suppose that $M = 2N+1$ for some $N \in \mathbb{N}$, and that $\ell^\infty(P_{2N+1})$ is C-projective in $\ell^1(P_{2N+1})$-**mod**. There is an isometric identification of Banach algebras $\ell^1(P_{2N+1}) = \ell^1(P_{2N}) \widehat{\otimes} \ell^1(P_1)$. By Proposition 2.2 $\ell^\infty(P_{2N})$ is C-projective in $\ell^1(P_{2N})$-**mod**. Therefore $C \ge 2^N = 2^{\lfloor M/2 \rfloor}$. □

Remark 4.7. The definition y in Theorem 4.6 does not depend on ρ. Hence we have shown that there exists $y \in \ell^\infty(P_M)$, such that

$$\inf_\rho \|\rho(y)\| \ge 2^{\lfloor M/2 \rfloor} ,$$

where the infimum is taken over all left $\ell^1(P_M)$-module morphisms $\rho : \ell^\infty(P_M) \to \ell^1(P_M) \widehat{\otimes} \ell^\infty(P_M)$ with $\pi \circ \rho = I_{\ell^\infty(P_M)}$.

Theorem 4.8. *Let X be an infinite set. Then the module $\ell^1(P_X)$ is not injective in* **mod**-$\ell^1(P_X)$*, and the module $\ell^1(F_X)$ is not injective in* **mod**-$\ell^1(F_X)$.

Proof. Assume towards a contradiction that $\ell^1(P_X)$ is C-right injective for some $C > 1$. Take $N \in \mathbb{N}$ and a subset $X_N \subset X$ with $|X_N| = 2N$. There is an isometric identification of Banach algebras $\ell^1(P_X) = \ell^1(P_N) \widehat{\otimes} \ell^1(P_{X \setminus X_N})$. By Proposition 2.2 $\ell^1(P_{2N})$ is C-right injective in **mod**-$\ell^1(P_{2N})$. By Theorem 4.6, $2^N \le C$. This is true for each $N \in \mathbb{N}$, the required contradiction.

The same argument, but using the identification $\ell^1(F_X) = \ell^1(P_N) \widehat{\otimes} \ell^1(F_{X \setminus X_N})$, shows that $\ell^1(F_X)$ is not injective in **mod**-$\ell^1(F_X)$. □

We find the following estimate involving alternating series interesting and difficult to prove directly.

Corollary 4.9. *Let $M \in \mathbb{N}$. Then*

$$2^{\lfloor 3M/2 \rfloor} \le \sup_\gamma \sum_{s \in P_M} \left| \sum_{t \in P_M} \gamma_t (-1)^{t \cap s} \right| \le 2^{3M/2} ,$$

where the supremum is taken over all complex sequences $\gamma = \{\gamma_t : t \in P_M\} \subset \overline{\mathbb{D}}$.

Proof. Let ρ be the map defined in Theorem 3.4. The proof of Theorem 3.4 shows that for $y = \sum_{t\in P_M}(-1)^{|t|}\gamma_t\lambda_t \in \ell^\infty(P_M)$ we have

$$\|\rho(y)\| = \frac{1}{2^M}\sum_{s\in P_M}\left|\sum_{t\in P_M}\gamma_t(-1)^{t\cap s}\right| .$$

We have shown in Theorems 3.4 and 4.6 that $2^{\lfloor M/2\rfloor} \le \|\rho\| \le 2^{M/2}$. The result follows. □

5 Open questions

We close with a brief statement of some open problems. The main question we leave open is the following.

Question 5.1. *For which semigroups S is $\ell^1(S)$ an injective right $\ell^1(S)$-module?*

The main result of this paper and the results of [6] strongly suggest that $\ell^1(S)$ is not an injective right module for any infinite semilattice, we *conjecture* that this is the case.

If S is weakly right cancellative (equivalently, $\ell^1(S)$ is a dual module) and $\ell^1(S)$ is amenable, then by standard theory $\ell^1(S)$ is an injective right $\ell^1(S)$-module. The natural conjecture: that these conditions are necessary, is false; in [6, Example 4.12] an example is given of a finite semigroup S such that $\ell^1(S)$ is an injective right module, but $\ell^1(S)$ is not amenable. The author does not know of any example of a non-weakly cancellative semigroup S such that $\ell^1(S)$ is an injective right module.

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