# Biflatness of semigroup algebras

Paul Ramsden

December 3, 2008

**Abstract**

We shall study the biflatness of the convolution algebra $\ell^1(S)$ for a semigroup $S$. We show that for any semigroup $S$ such that $\ell^1(S)$ is biflat the canonical partial ordering on the idempotents must be uniformly locally finite. We use this to characterize the biflatness of $\ell^1(S)$ for an inverse semigroup $S$.

## 1 Introduction

In [2] the biflatness of the Banach algebra $\ell^1(S)$ was characterized for a Clifford semigroup $S$. Indeed the following theorem was proved.

**Theorem 1.1** ([2, Theorem 6.1])**.** *Let $S$ be a Clifford semigroup. Then $\ell^1(S)$ is biflat if and only if:*

(i) $(E(S), \leq)$ *is uniformly locally finite; and*

(ii) *each maximal subgroup is amenable.* □

The proof of this theorem uses the representation theory for inverse semigroups developed in [14] and [15], but adapted to the Banach algebra setting.

We shall prove in Theorem 3.4 that, for any semigroup $S$ such that $\ell^1(S)$ is biflat, $(E(S), \leq)$ is uniformly locally finite. This is similar to a theorem of Duncan and Paterson [6, Theorem 2] ($\ell^1(S)$ amenable $\Rightarrow E(S)$ finite). This theorem allows us to extend the method of [2] to characterize the biflatness and biprojectivity of $\ell^1(S)$ for the class of inverse semigroups (Theorem 3.7).

A biflat Banach algebra is weakly amenable [3, Proposition 2.8.62]. Hence our results provide further examples of non-commutative Banach algebras that are weakly amenable.

## 2 Preliminaries

### 2.1 Banach algebras

For the background theory of modules over a Banach algebra, see [3] and [11]. For Banach spaces $E$ and $F$ the Banach space of all bounded linear operators from $E$ to $F$ is denoted by $\mathcal{B}(E,F)$. We set $E' = \mathcal{B}(E,\mathbb{C})$; the action of $\lambda \in E'$ on an element $x \in E$ is written as $\langle x, \lambda \rangle$. For $r > 0$ we set $E_{[r]} = \{x \in E : \|x\| \leq r\}$.

Let $A$ and $B$ be Banach algebras. Then the space $A \widehat{\otimes} B$ becomes a Banach algebra with the multiplication given by

$$(a_1 \otimes b_1)(a_2 \otimes b_2) = a_1 a_2 \otimes b_1 b_2 \quad (a_1, a_2 \in A,\ b_1, b_2 \in B)\,.$$

Let $A$ be a Banach algebra, and let $\Lambda$ be a non-empty set. We denote by $\mathbb{M}_\Lambda(A)$ be the set of $\Lambda \times \Lambda$ matrices $(a_{ij})_{i,j\in\Lambda}$ with entries in $A$ such that

$$\|(a_{ij})\| = \sum_{i,j} \|a_{ij}\|_A < \infty\,.$$

Then $\mathbb{M}_\Lambda(A)$ is a Banach algebra with matrix multiplication. This Banach algebra belongs to the class of $\ell^1$-*Munn algebras* introduced in [7]. The *matrix units* in $\mathbb{M}_\Lambda(\mathbb{C})$ are denoted by $E_{i,j}$, so that

$$E_{i,j} E_{k,l} = \delta_{j,k} E_{i,l} \quad (i,j,k,l \in \Lambda)\,,$$

where $\delta_{j,k} = 1$ if $j = k$ and $\delta_{j,k} = 0$ if $j \neq k$. The map

$$\theta : (a_{i,j}) \mapsto \sum_{i,j} a_{i,j} \otimes E_{i,j}, \quad \mathbb{M}_\Lambda(A) \to A \widehat{\otimes} \mathbb{M}_\Lambda(\mathbb{C})\,,$$

is an isometric algebra isomorphism.

Let $\{A_\lambda : \lambda \in \Lambda\}$ be a collection of Banach algebras. Then the $\ell^1$-direct sum

$$\ell^1\text{-}\bigoplus\{A_\lambda : \lambda \in \Lambda\}$$

is a Banach algebra with respect to componentwise multiplication, where multiplication in the $\lambda^{\text{th}}$ component is just multiplication in $A_\lambda$.

Let $A$ be a Banach algebra, and let $E$ be a Banach $A$-bimodule. We consider the dual space $E'$ as an $A$-bimodule with the multiplication

$$\langle x, \lambda \cdot a \rangle = \langle a \cdot x, \lambda \rangle \quad \text{and} \quad \langle x, a \cdot \lambda \rangle = \langle a \cdot x, \lambda \rangle \quad (x \in E, \lambda \in E', a \in A)\,.$$

Let $A$ be a Banach algebra. Then the projective tensor product $A \widehat{\otimes} A$ is a Banach $A$-bimodule, where the multiplication is specified by

$$a \cdot (b \otimes c) = ab \otimes c \quad \text{and} \quad (b \otimes c) \cdot a = b \otimes ca \quad (a, b, c \in A).$$

We define the *multiplication map* $\pi : A \widehat{\otimes} A \to A$ by

$$\pi(a \otimes b) = ab \quad (a, b \in A).$$

Quantitative versions of standard homological properties were first introduced and studied in [16].

**Definition 2.1.** Let $A$ be a Banach algebra, and let $C > 0$. Then $A$ is:

- *$C$-biprojective* if there exists an $A$-bimodule morphism $\rho : A \to A \widehat{\otimes} A$ with $\pi \circ \rho = I_A$ and $\|\rho\| \leq C$;
- *$C$-biflat* if there exists an $A$-bimodule morphism $\rho : (A \widehat{\otimes} A)' \to A'$ with $\rho \circ \pi' = I_{A'}$ and $\|\rho\| \leq C$;
- *$C$-amenable* if there exists a bounded net $(d_\alpha) \subset A \widehat{\otimes} A$ with

$$a \cdot d_\alpha - d_\alpha \cdot a \to 0, \qquad a\pi(d_\alpha) \to a \quad (a \in A)$$

  and $\sup \|d_\alpha\| \leq C$.

For an amenable or biflat Banach algebra $A$ we set

$$AM(A) = \inf \{C > 0 : A \text{ is } C\text{-amenable}\},$$

and

$$BF(A) = \inf \{C > 0 : A \text{ is } C\text{-biflat}\},$$

respectively. It is known that a Banach algebra $A$ is amenable if and only if $A$ is biflat and has a bounded approximate identity [3, Theorem 2.9.65], in which case we have the following relationship between the associated constants

$$BF(A) \leq AM(A) \leq BF(A) \cdot \inf \left\{ \sup_\alpha \|e_\alpha\| : (e_\alpha) \text{ is a bai for } A \right\}.$$

The following is a useful characterization of biflatness.

**Proposition 2.2** ([11, Exercise VII2.8])**.** *Let $A$ be a Banach algebra, and let $C > 0$. Then $A$ is $C$-biflat if and only if there exists an $A$-bimodule morphism $\rho$ such that the diagram*

$$\begin{array}{ccc} & & (A \widehat{\otimes} A)'' \\ & \overset{\rho}{\nearrow} & \downarrow \pi'' \\ A & \xrightarrow{\;i_A\;} & A'' \end{array}$$

*commutes and $\|\rho\| \leq C$. Here $i_A : A \to A''$ is the natural embedding into the second dual.* □

The proof of [2, Proposition 6.3] easily gives the following.

**Proposition 2.3.** *Let $\{A_\lambda : \lambda \in \Lambda\}$ be a collection of Banach algebras. Then $\ell^1$-$\bigoplus\{A_\lambda : \lambda \in \Lambda\}$ is biflat [biprojective] if and only if there is a constant $C > 0$ such that each $A_\lambda$ $(\lambda \in \Lambda)$ is $C$-biflat [$C$-biprojective].* □

**Proposition 2.4.** *Let $A$ be a $C_1$-biprojective Banach algebra, and let $B$ be $C_2$-biprojective Banach algebra. Then $A \widehat{\otimes} B$ is $C_1C_2$-biprojective.*

*Proof.* There exist an $A$-bimodule morphism $\rho_A : A \to A \widehat{\otimes} A$ with $\pi_A \circ \rho_A = I_A$ and $\|\rho_A\| \leq C_1$ and a $B$-bimodule morphism $\rho_B : B \to B \widehat{\otimes} B$ with $\pi_B \circ \rho_B = I_B$ and $\|\rho_B\| \leq C_2$. Let $\theta : (A \widehat{\otimes} A) \widehat{\otimes} (B \widehat{\otimes} B) \to (A \widehat{\otimes} B) \widehat{\otimes} (A \widehat{\otimes} B)$ be the isometric isomorphism given by

$$a_1 \otimes a_2 \otimes b_1 \otimes b_2 \mapsto a_1 \otimes b_1 \otimes a_2 \otimes b_2 \quad (a_1, a_2 \in A,\ b_1, b_2 \in B)\,.$$

We set

$$\rho = \theta \circ (\rho_A \otimes \rho_B) : A \widehat{\otimes} B \to (A \widehat{\otimes} B) \widehat{\otimes} (A \widehat{\otimes} B)\,.$$

It is easily checked that $\rho$ is an $A \widehat{\otimes} B$-bimodule morphism. Since $\pi_{A \widehat{\otimes} B} = (\pi_A \otimes \pi_B) \circ \theta^{-1}$, we have $\pi_{A \widehat{\otimes} B} \circ \rho = I_A \otimes I_B = I_{A \widehat{\otimes} B}$. Further $\|\rho\| \leq C_1C_2$. Therefore $A \widehat{\otimes} B$ is $C_1C_2$-biprojective. □

Let $A$ and $B$ be Banach algebras, let $E$ be a Banach $A$-bimodule, and let $F$ be a Banach $B$-bimodule. We regard $E \widehat{\otimes} F$ is a Banach $A \widehat{\otimes} B$-bimodule with the multiplication given by

$$\left.\begin{array}{l} (a \otimes b) \cdot (x \otimes y) = (a \cdot x) \otimes (b \cdot y) \\ (x \otimes y) \cdot (a \otimes b) = (x \cdot a) \otimes (y \cdot b) \end{array}\right\} \quad (a \in A,\ b \in B,\ x \in E, y \in F)\,.$$

We also regard $\mathcal{B}(E,F)$ as a Banach $A \widehat{\otimes} B$-bimodule with the multiplication given by

$$\left.\begin{aligned} ((a\otimes b)*T)(x) &= b\cdot T(x\cdot a)\\ (T*(a\otimes b))(x) &= T(a\cdot x)\cdot b \end{aligned}\right\} \quad (T\in\mathcal{B}(E,F),\, a\in A,\, b\in B,\, x\in E)\,.$$

We denote this module by $\widetilde{\mathcal{B}}(E,F)$. Similarly $\mathcal{B}(F,E)$ is a Banach $A \widehat{\otimes} B$-bimodule with the multiplication given by

$$\left.\begin{aligned} ((a\otimes b)*T)(x) &= a\cdot T(x\cdot b)\\ (T*(a\otimes b))(x) &= T(b\cdot x)\cdot a \end{aligned}\right\} \quad (T\in\mathcal{B}(F,E),\, a\in A,\, b\in B,\, x\in E)\,.$$

We denote this module by $\widehat{\mathcal{B}}(F,E)$. Let $\lambda \in (E \widehat{\otimes} F)'$. Then we define $\widetilde{T_\lambda} \in \mathcal{B}(E,F')$ and $\widehat{T_\lambda} \in \mathcal{B}(F,E')$ by

$$\left.\begin{aligned} \left\langle y, \widetilde{T_\lambda}(x)\right\rangle &= \langle x\otimes y, \lambda\rangle\\ \left\langle x, \widehat{T_\lambda}(y)\right\rangle &= \langle x\otimes y, \lambda\rangle \end{aligned}\right\} \quad (x\in E,\, y\in F)\,.$$

The maps

$$\lambda \mapsto \widetilde{T_\lambda}, \quad (E\widehat{\otimes} F)' \to \widetilde{\mathcal{B}}(E,F') \quad \text{and} \quad \lambda \mapsto \widehat{T_\lambda}, \quad (E\widehat{\otimes} F)' \to \widehat{\mathcal{B}}(E,F')$$

are isometric $A \widehat{\otimes} B$-bimodule isomorphisms.

**Proposition 2.5.** *Let $A$ be a $C_1$-biflat Banach algebra, and let $B$ be a $C_2$-biflat Banach algebra. Then $A \widehat{\otimes} B$ is $C_1C_2$-biflat.*

*Proof.* There exist an $A$-bimodule morphism $\rho_A : (A\widehat{\otimes} A)' \to A'$ with $\rho_A \circ \pi'_A = I_A$ and $\|\rho_A\| \le C_1$ and a $B$-bimodule morphism $\rho_B : (B\widehat{\otimes} B)' \to B'$ with $\rho_B \circ \pi'_B = I_{B'}$ and $\|\rho_B\| \le C_2$.

Now consider the composition

$$\begin{aligned} \rho_{A\widehat{\otimes} B} : ((A\widehat{\otimes} B)\widehat{\otimes}(A\widehat{\otimes} B))' \cong \widetilde{\mathcal{B}}(A\widehat{\otimes} A, (B\widehat{\otimes} B)') &\overset{T\mapsto\rho_B\circ T}{\longrightarrow} \widetilde{\mathcal{B}}(A\widehat{\otimes} A, B')\\ \cong \widehat{\mathcal{B}}(B,(A\widehat{\otimes} A)') \overset{T\mapsto\rho_A\circ T}{\longrightarrow} \widehat{\mathcal{B}}(B,A') &\cong (A\widehat{\otimes} B)'\,. \end{aligned}$$

It is immediately checked that each of the maps is an $A \widehat{\otimes} B$-bimodule morphism, and that $\left\|\rho_{A\widehat{\otimes} B}\right\| \le C_1C_2$. Take $\lambda \in (A\widehat{\otimes} B)'$. We follow $\pi'_{A\widehat{\otimes} B}(\lambda)$ under the sequence of compositions $\rho_{A\widehat{\otimes} B}$. We have

$$\pi'_{A\widehat{\otimes} B}(\lambda) \mapsto \pi'_B \circ \widetilde{T_\lambda} \circ \pi_A \mapsto \widetilde{T_\lambda} \circ \pi_A \mapsto \pi'_A \circ \widehat{T_\lambda} \mapsto \widehat{T_\lambda} \mapsto \lambda\,.$$

Hence $\rho_{A\widehat{\otimes} B} \circ \pi'_{A\widehat{\otimes} B} = I_{(A\widehat{\otimes} B)'}$, and therefore $A \widehat{\otimes} B$ is $C_1C_2$-biflat. ∎

We now prove a partial converse to Propositions 2.4 and 2.5. For a similar result about amenability, see [13, Proposition 3.5].

**Proposition 2.6.** *Let $A$ be a unital Banach algebra, and let $B$ be a Banach algebra containing a non-zero idempotent $b_0$. Suppose that $A \widehat{\otimes} B$ is $C$-biflat [$C$-biprojective]. Then $A$ is $C\,\|b_0\|$-biflat [$C\,\|b_0\|$-biprojective].*

*Proof.* First suppose that $A \widehat{\otimes} B$ is $C$-biprojective. Then there exists an $A \widehat{\otimes} B$-bimodule morphism $\rho : A \widehat{\otimes} B \to (A \widehat{\otimes} B) \widehat{\otimes} (A \widehat{\otimes} B)$ with $\pi_{A \widehat{\otimes} B} \circ \rho = I_{A \widehat{\otimes} B}$ and $\|\rho\| \leq C$. We regard $A \widehat{\otimes} B$ as an $A$-bimodule with the actions given by

$$a_1 \cdot (a_2 \otimes b) = a_1 a_2 \otimes b, \quad \text{and} \quad (a_2 \otimes b) \cdot a_1 = a_2 a_1 \otimes b \quad (a_1, a_2 \in A, b \in B)\,.$$

Then we have

$$\begin{aligned} \rho(a_1 a_2 \otimes b_0) &= \rho((a_1 \otimes b_0)(a_2 \otimes b_0)) = (a_1 \otimes b_0) \cdot \rho(a_2 \otimes b_0) \\ &= a_1 \cdot (e_A \otimes b_0) \cdot \rho(a_2 \otimes b_0) = a_1 \cdot \rho(a_2 \otimes b_0) \quad (a_1, a_2 \in A)\,. \end{aligned}$$

Similarly we can show a right-module version of this equation. Hence we have

$$\rho(a_1 a_2 \otimes b_0) = a_1 \cdot \rho(a_2 \otimes b_0) = \rho(a_1 \otimes b_0) \cdot a_2 \quad (a_1, a_2 \in A)\,. \tag{2.1}$$

Take $\varphi \in (B')_{[1]}$ with $\langle b_0, \varphi \rangle = 1$ and define

$$\theta : (a_1 \otimes b_1) \otimes (a_2 \otimes b_2) \mapsto \varphi(b_1 b_2) a_1 \otimes a_2, \quad (A \widehat{\otimes} B) \widehat{\otimes} (A \widehat{\otimes} B) \to A \widehat{\otimes} A\,.$$

Then $\theta$ is an $A$-bimodule morphism.

We now define $\widetilde{\rho} : A \to A \widehat{\otimes} A$ by

$$\widetilde{\rho}(a) = \theta \circ \rho(a \otimes b_0) \quad (a \in A)\,.$$

By (2.1), $\widetilde{\rho}$ is an $A$-bimodule morphism. It follows from the identity

$$\pi_A \circ \theta = (I_A \otimes \varphi) \circ \pi_{A \widehat{\otimes} B}$$

that $\pi_A \circ \widetilde{\rho} = I_A$. Further $\|\widetilde{\rho}\| \leq C\,\|b_0\|$. Therefore $A$ is $C\,\|b_0\|$-biprojective.

The proof for biflatness is similar. Given $\rho : A \widehat{\otimes} B \to ((A \widehat{\otimes} B) \widehat{\otimes} (A \widehat{\otimes} B))''$ with $\pi''_{A \widehat{\otimes} B} \circ \rho = i_{A \widehat{\otimes} B}$, we define $\widetilde{\rho} : A \to (A \widehat{\otimes} A)''$ by

$$\widetilde{\rho}(a) = \theta'' \circ \rho(a \otimes b_0) \quad (a \in A)\,.$$

It is easily checked that $\widetilde{\rho}$ has the required properties. □

**Proposition 2.7.** *Let $A$ be a unital Banach algebra, and let $\Lambda$ be a non-empty set. Then $\mathbb{M}_\Lambda(A)$ is $C$-biflat [$C$-biprojective] if and only if $A$ is $C$-biflat [$C$-biprojective].*

*Proof.* Set $B = \mathbb{M}_\Lambda(\mathbb{C})$. First note that $B$ is 1-biprojective. Indeed, fix $k_0 \in \Lambda$, and define $\rho : B \to B \widehat{\otimes} B$ by

$$\rho(a) = \sum_{i,j \in \Lambda} a_{ij} E_{ik_0} \otimes E_{k_0 j} \quad (a = (a_{ij}) \in B) .$$

The sum converges since $\sum_{i,j} |a_{ij}| < \infty$. It is easily checked on matrix units that $\rho$ is a $B$-bimodule morphism which is a right inverse to the multiplication map.

Although $B$ is not unital if the index set $\Lambda$ is infinite, $B$ always contains non-zero idempotents of norm 1. Since $\mathbb{M}_\Lambda(A) = A \widehat{\otimes} B$, the result follows from the preceding propositions. $\square$

The analogous result for '$C$-amenable' is not true. See [4, Theorem 2.7] for a relation between $AM(\mathbb{M}_\Lambda(A))$ and $AM(A)$.

## 2.2 Semigroups

Our standard reference for the algebraic theory of semigroups is [12]. We shall use the following notation introduced by Grønbæk in [9]. For $s, t \in S$ we define the sets

$$[st^{-1}] = \{u \in S : ut = s\} , \tag{2.2}$$
$$[t^{-1}s] = \{u \in S : tu = s\} . \tag{2.3}$$

The study of ideals generated by elements in a semigroup leads to several important equivalence relations on a semigroup known as *Green's equivalences*. The equivalence relation that will be important to us is the following.

**Definition 2.8.** Let $S$ be a semigroup, and let $s, t \in S$. We define a relation $\mathcal{D}$ on $S$ by $s \,\mathcal{D}\, t$ if and only if there exists $x \in S$ with $Ss \cup \{s\} = Sx \cup \{x\}$ and $xS \cup \{x\} = tS \cup \{t\}$.

For a proof that this is an equivalence relation see [12, Chapter 2].

An element $p \in S$ is *idempotent* if $p^2 = p$. The set of idempotents is denoted by $E(S)$. A semigroup $S$ is a *semilattice* if $S$ is commutative and $E(S) = S$. The *canonical partial order* on $E(S)$ is given by

$$p \leq q \Longleftrightarrow p = pq = qp \quad (p, q \in E(S)) .$$

An idempotent $p$ is *maximal* if $p = q$ whenever $p \leq q$.

Let $S$ be a semigroup, and let $s \in S$. An element $s^* \in S$ is an *inverse* of $s$ if

$$ss^*s = s \quad \text{and} \quad s^*ss^* = s^* .$$

In general an inverse will not be unique. An element $s \in S$ is *regular* if there exists $t \in S$ with $sts = s$. Clearly if $s$ has an inverse then $s$ is regular; less obviously the converse is also true. Indeed suppose that there is $t \in S$ with $sts = s$. Set $s^* = tst$. Then we have

$$ss^*s = (sts)ts = sts = s \text{ and } s^*ss^* = t(sts)tst = t(sts)t = s^* .$$

The set of regular elements of $S$ is denoted by $R(S)$. The semigroup $S$ is *regular* if $S = R(S)$. An element that is not regular is called *singular*, and the set of singular elements is denoted by $N(S)$.

**Definition 2.9.** Let $S$ be a semigroup. Then $S$ is an *inverse semigroup* if $S$ is regular and every element has a unique inverse.

We shall denote the inverse of an element $s$ in an inverse semigroup by $s^{-1}$. By [12, Proposition 5.1.1] a semigroup $S$ is an inverse semigroup if and only if $S$ is regular and the idempotents commute.

Let $S$ be an inverse semigroup. There is a natural partial order on $S$ given by $s \leq t \Leftrightarrow s = ss^{-1}t$. This agrees with the usual partial order on $E(S)$.

*Remark* 2.10. There are various equivalent definitions of the partial order on $S$; see [12, Proposition 5.2.1]. For example $s \leq t \Leftrightarrow s = ts^{-1}s \Leftrightarrow s = tp$ for some $p \in E(S)$.

We shall use the following characterization of $\mathcal{D}$-classes in an inverse semigroup.

**Proposition 2.11** ([12, Proposition 5.1.2(4)])**.** *Let $S$ be an inverse semigroup, and let $s, t \in S$. Then $s \,\mathcal{D}\, t$ if and only if there exists $x \in S$ with $s^{-1}s = xx^{-1}$ and $t^{-1}t = x^{-1}x$.* □

Let $S$ be an inverse semigroup, and let $p \in E(S)$. We set

$$G_p = \{s \in S : ss^{-1} = s^{-1}s = p\}\,.$$

Then $G_p$ is a group with identity $p$, and $G_p$ contains any other subgroup of $S$ with identity $p$. Thus $G_p$ is called the *maximal subgroup of $S$ at $p$*. Suppose that idempotents $p$ and $q$ lie in the same $\mathcal{D}$-class. Take $x \in S$ with $p = xx^{-1}$ and $q = x^{-1}x$. Then the map

$$s \mapsto x^{-1}sx, \quad G_p \to G_q\,,$$

is an isomorphism.

**Definition 2.12.** Let $S$ be a semigroup. Then $S$ is a *Clifford semigroup* if $S$ is an inverse semigroup such that

$$ss^{-1} = s^{-1}s \quad (s \in S)\,.$$

Let $P$ be a partially ordered set. For $p \in P$, we define $(p] = \{x : x \leq p\}$ and $[p) = \{x : p \leq x\}$. Then $P$ is *locally finite* if $(p]$ is finite for each $p \in P$, and $P$ is *locally $C$-finite* for some constant $C > 1$ if $|(p]| < C$ for each $p \in P$. A partially ordered set that is locally $C$-finite for some $C$ is *uniformly locally finite.*

**Definition 2.13.** Let $S$ be an inverse semigroup. Then $S$ is [*locally finite* / *$C$-locally finite* / *uniformly locally finite*] respectively if the partially ordered set $(E(S), \leq)$ has the corresponding property.

**Proposition 2.14.** *Let $S$ be an inverse semigroup. Suppose that $(E(S), \leq)$ is [uniformly] locally finite. Then $(S, \leq)$ is [uniformly] locally finite.*

*Proof.* For $t \in S$ we set $(t]_S = \{s \in S : s \leq t\}$ and $(t]_E = \{s \in E(S) : s \leq t\}$. For each $p \in E(S)$ we have $(p]_S = (p]_E$. Fix $t \in S$. The inclusion $\{tp : p \in (t^{-1}t]_E\} \subset (t]_S$ is clear. Now take $s \leq t$. Then $s = ts^{-1}s = t(t^{-1}ts^{-1}s)$ and $t^{-1}ts^{-1}s \leq t^{-1}t$, hence

$$(t]_S = \{tp : p \in (t^{-1}t]_E\}\,.$$

The result follows. □

## 2.3 Semigroup algebras

Let $S$ be a semigroup. The *semigroup algebra* $\ell^1(S)$ is the completion in the $\ell^1$-norm of the algebra $\mathbb{C}S$. It is the *Banach algebra generated by the semigroup.* For $s \in S$ we write $\delta_s = \chi_{\{s\}}$ for the indicator function of the set $\{s\}$. The *convolution product* $\star$ on $\ell^1(S)$ is uniquely defined by requiring that $\delta_s \star \delta_t = \delta_{st}$ $(s, t \in S)$. These Banach algebras have been studied by many authors. A recent memoir is [4], which contains many references to original papers.

### 2.3.1 The Banach algebra $\ell^1(S)$ for a ULF inverse semigroup

We shall show that for a uniformly locally finite (ULF) inverse semigroup the Banach algebra $\ell^1(S)$ is isomorphic to a direct sum of $\ell^1$-Munn algebras over group algebras. This an adaption to the Banach algebra setting of [15, Theorem 4.6] using [2, Proposition 6.5]. Since the terminology of [15] is different from ours and to aid the reader we give some of the details.

Let $S$ be an inverse semigroup. We define a multiplication $*$ on the space $\ell^1(S)$ by

$$\delta_s * \delta_t = \begin{cases} \delta_{st}, & \text{if } s^{-1}s = tt^{-1} \\ 0, & \text{otherwise} \end{cases} .$$

A direct check shows that this defines an associative product. We denote the resulting Banach algebra by $B(S)$. Let $D$ be a $\mathcal{D}$-class of $S$. We set $B(D) = \ell^1(D)$, regarded as a subalgebra of $B(S)$.

**Proposition 2.15.** *Let $S$ be an inverse semigroup.*

(i) *$B(D)$ is an ideal in $B(S)$ for each $\mathcal{D}$-class $D$.*

(ii) *Let $\{D_\lambda : \lambda \in \Lambda\}$ be the family of $\mathcal{D}$-classes of $S$ indexed by some set $\Lambda$. Then there is an isometric isomorphism of Banach algebras*

$$B(S) \cong \ell^1\text{-}\bigoplus\{B(D_\lambda) : \lambda \in \Lambda\} .$$

*Proof.* (i) Fix a $\mathcal{D}$-class $D$. Take $s \in S$ and $t \in D$. If $\delta_s * \delta_t = 0$, then $\delta_s * \delta_t \in B(D)$. Otherwise $\delta_s * \delta_t = \delta_{st}$ and $s^{-1}s = tt^{-1}$. Then we have

$$(st)^{-1}(st) = t^{-1}s^{-1}st = t^{-1}tt^{-1}t = t^{-1}t .$$

Setting $x = t^{-1}t$ in Proposition 2.11 we see that $st\,\mathcal{D}\,t$. Hence $\delta_s * \delta_t \in B(D)$ and $B(D)$ is a left ideal. Similarly it is a right ideal.

(ii) Clearly there is an isomorphism of Banach spaces. We just need to show that, if $s$ and $t$ lie in different $\mathcal{D}$-classes, then $\delta_s * \delta_t = 0$. Suppose that $\delta_s * \delta_t \neq 0$. Then $s^{-1}s = tt^{-1}$ and so $s \,\mathcal{D}\, t^{-1}$. Since $t^{-1} \,\mathcal{D}\, t$ we have $s \,\mathcal{D}\, t$. $\square$

Let $P$ be a partially ordered set. Following the notation of [2] we define $\mathrm{Sch} : \ell^1(P) \to \ell^\infty(P)$ by

$$\mathrm{Sch}(\delta_t) = \chi_{(t]} \quad (t \in S)\,,$$

where $\chi_{(t]}$ is the indicator function for the set $(t]$. The map Sch is called the *Schützenburger representation* of $P$; see [2, §4] and the references therein. Suppose that $P$ is uniformly locally finite. Then by [2, Proposition 6.5] the range of Sch is contained in $\ell^1(P)$ and $\mathrm{Sch} : \ell^1(P) \to \ell^1(P)$ is an isomorphism of Banach spaces.

**Theorem 2.16** (*cf.* [15, Theorem 4.5])**.** *Let $S$ be a ULF inverse semigroup with $\mathcal{D}$-classes $\{D_\lambda : \lambda \in \Lambda\}$. Then there is an isomorphism of Banach algebras*

$$\ell^1(S) \cong \ell^1\text{-}\bigoplus\{B(D_\lambda) : \lambda \in \Lambda\}\,.$$

*Proof.* Let $B(S)$ be the Banach algebra defined above. By [2, Proposition 6.5] the Schützenburger representation $\mathrm{Sch} : \ell^1(S) \to B(S)$ is an isomorphism of Banach spaces. It is proved in [15, Lemma 4.1] that Sch is an algebra homomorphism. The result now follows from Proposition 2.15. $\square$

We now identify the Banach algebra $B(D)$ for a $\mathcal{D}$-class $D$.

**Theorem 2.17.** *Let $S$ be an inverse semigroup, and let $D \subset S$ be a $\mathcal{D}$-class. Take $\widetilde{p} \in E(D)$. Then there is an isometric isomorphism of Banach algebras*

$$B(D) \cong \mathbb{M}_{E(D)}(\ell^1(G_{\widetilde{p}}))\,.$$

*Proof.* Set $G = G_{\widetilde{p}}$ which, up to isomorphism, does not depend on the choice of $\widetilde{p}$. For $p, q \in E(D)$ define

$$X_{p,q} = \{s \in D : ss^{-1} = p,\ s^{-1}s = q\}\,.$$

We have $X_{p,p} = G_p$. For each $p \in E(D)$ we use Proposition 2.11 to choose $x_p \in S$ with $\widetilde{p} = x_p x_p^{-1}$ and $p = x_p^{-1} x_p$. Then the map

$$\theta_{p,q} : s \mapsto x_p s x_q^{-1}, \quad X_{p,q} \to G\,,$$

extends to an isometric isomorphism of Banach spaces $\theta_{p,q} : \ell^1(X_{p,q}) \to \ell^1(G)$. Let $P_{p,q} : B(D) \to \ell^1(X_{p,q})$ be the natural projection. Define

$$\theta : B(D) \to \mathbb{M}_{E(D)}(\ell^1(G))$$

by

$$\theta(a) = \sum_{p,q \in E(D)} \theta_{p,q} \circ P_{p,q}(a) E_{p,q} \quad (a \in B(D))\,.$$

Then $\theta$ is an isometric isomorphism of Banach spaces. For $p, q, u, v \in E(D)$ we have

$$\ell^1(X_{p,q}) * \ell^1(X_{u,v}) = 0 \text{ if } q \neq u, \text{ and } \ell^1(X_{p,q}) * \ell^1(X_{q,v}) = \ell^1(X_{p,v})\,.$$

From this it follows that $\theta$ is a Banach algebra isomorphism. □

**Theorem 2.18** (*cf.* [15, Theorem 4.6])**.** *Let $S$ be a ULF inverse semigroup with $\mathcal{D}$-classes $\{D_\lambda : \lambda \in \Lambda\}$. For each $\lambda$ take an idempotent $p_\lambda \in D_\lambda$. Then there is an isomorphism of Banach algebras*

$$\ell^1(S) \cong \ell^1\text{-}\bigoplus \{\mathbb{M}_{E(D_\lambda)}(\ell^1(G_{p_\lambda})) : \lambda \in \Lambda\}\,.$$

*Proof.* This follows by combining Theorem 2.16 and Theorem 2.17. □

Suppose that $S$ is a Clifford semigroup. Then each $\mathcal{D}$-class $D$ contains a single idempotent $p$ (say) and $B(D) = \ell^1(G_p)$.

**Proposition 2.19.** *Let $S$ be an inverse semigroup. Suppose that either:*

(i) *$S$ is locally finite and $\ell^1(S)$ has an identity, or*

(ii) *$S$ is uniformly locally finite and $\ell^1(S)$ has a bounded approximate identity.*

*Then $E(S)$ is finite.*

*Proof.* (i) By [5, Lemma 13] $\ell^1(S)$ has an identity if and only if $\ell^1(E(S))$ has an identity. Therefore by Proposition 2.14 it is sufficient to prove the result in the case where $S$ is a semilattice, and so we suppose that this is the case.

Let $e_A = \sum_{s \in S} e_s \delta_s$ be an identity for $\ell^1(S)$. For each $t \in S$, the equation $\delta_t \star e_A = \delta_t$ implies that

$$\sum_{s \in [t)} e_s = 1\,.$$

Let $M$ be the set of maximal elements of $S$. For each $m \in M$ we have $[m) = \{m\}$. It follows that there are only finitely many maximal elements.

There is a finite set $F \subset S$ such that

$$\sum_{s \in S \setminus F} |e_s| < 1/2 \, .$$

Take $t \in S$ and assume towards a contradiction that there exists an infinite increasing chain

$$t < s_1 < s_2 < \cdots$$

in $S$. Let $f \in F$. Since $S$ is locally finite, the inequality $s_n \leq f$ holds for only finitely many $n \in \mathbb{N}$. Set

$$N = \max_{f \in F} \{\max \{n \in \mathbb{N} : f \in [s_n)\}\} \, .$$

Then $F \cap [s_{N+1}) = \emptyset$. However $\sum_{s \in [s_{N+1})} e_s = 1$, which is a contradiction. Therefore, for each $t \in S$, there exists $m \in M$ with $t \leq m$. Hence $S \subset \bigcup_{m \in M} (m]$ and $S$ is finite.

(ii) This is similar to part (i). Again by [5, Lemma 13] we may suppose that $S$ is a semilattice, and an easy modification of the above argument shows that there are finitely many maximal elements. From the hypothesis that $S$ is uniformly locally finite, $S \subset \bigcup_{m \in M} (m]$. Therefore $S$ is finite. □

**Example 2.20.** Let $S = (\mathbb{N}, \min)$. Then $S$ is locally finite but not uniformly locally finite, and $\ell^1(S)$ has a bounded approximate identity, but not an identity.

# 3 Main results

Let $S$ be a semigroup, and set $A = \ell^1(S)$. Suppose that $A$ is biflat. Then by Proposition 2.2 there exists an $A$-bimodule morphism $\rho : A \to (A \widehat{\otimes} A)''$ with $\pi'' \circ \rho = i_A$. Fix $u \in S$. Suppose that $ru = vw$ for some elements $r, v, w \in S$, and set $\theta = ru = vw$. We can find nets $(z_\alpha)$ and $(w_\alpha)$ in $(A \widehat{\otimes} A)_{[\|\rho\|]}$ indexed by the same directed set such that $\lim_\alpha z_\alpha = \rho(\delta_u)$ and $\lim_\alpha w_\alpha = \rho(\delta_v)$ in the weak-$*$ topology. Set $\lambda_\theta = \chi_{\{\theta\}} \in \ell^\infty(S) = A'$. Then we have

$$1 = \langle \pi'(\lambda_\theta), \rho(\delta_\theta) \rangle = \lim_\alpha \langle \pi'(\lambda_\theta), \delta_r \cdot z_\alpha \rangle = \lim_\alpha \langle \lambda_\theta, \pi(\delta_r \cdot z_\alpha) \rangle \, . \qquad (3.1)$$

Since $\lim_\alpha (\delta_r \cdot z_\alpha - w_\alpha \cdot \delta_w) = 0$ in the weak topology on $A \widehat{\otimes} A$, we may by Mazur's Theorem suppose that

$$\lim_\alpha \|\delta_r \cdot z_\alpha - w_\alpha \cdot \delta_w\|_\pi = 0\,. \tag{3.2}$$

Using the identification $A \widehat{\otimes} A = \ell^1(S \times S)$, for each $\alpha$ we can write $z_\alpha = \sum_{s,t\in S} z^\alpha_{s,t}\delta_{(s,t)}$ and $w_\alpha = \sum_{s,t\in S} w^\alpha_{s,t}\delta_{(s,t)}$ where $(z^\alpha_{s,t}), (w^\alpha_{s,t}) \subset \mathbb{C}$.

**Lemma 3.1.** *We have*

$$\lim_\alpha \sum_{(y,t)\in Z(r,w,\theta)} z^\alpha_{y,t} = 1\,, \tag{3.3}$$

*where* $Z(r,w,\theta) = \{(y,t) \in S \times S : t \in Sw, ryt = \theta\}$.

*Proof.* From equation (3.2) we have

$$0 = \lim_\alpha (\delta_r \cdot z_\alpha - w_\alpha \cdot \delta_w) = \lim_\alpha \sum_{s,t\in S} \left( \sum_{y\in[r^{-1}s]} z^\alpha_{y,t} - \sum_{x\in[tw^{-1}]} w^\alpha_{s,x} \right) \delta_{(s,t)}\,. \tag{3.4}$$

Since $S \setminus Sw = \{t \in S : [tw^{-1}] = \emptyset\}$, taking the norm of this expression and removing the summands for $t \in Sw$ gives

$$\lim_\alpha \sum_{s\in S} \sum_{t\in S\setminus Sw} \left| \sum_{y\in[r^{-1}s]} z^\alpha_{y,t} \right| = 0\,.$$

Removing the summands for $t \in S \setminus [s^{-1}\theta]$ gives

$$\lim_\alpha \sum_{s\in S} \sum_{t\in(S\setminus Sw)\cap[s^{-1}\theta]} \left| \sum_{y\in[r^{-1}s]} z^\alpha_{y,t} \right| = 0\,.$$

For each $s, t, y$ in the sum we have $st = \theta$ and $ry = s$ giving $ryt = \theta$. Hence

$$\lim_\alpha \sum_{\{(y,t)\in S\times S:\, t\in S\setminus Sw,\, ryt=\theta\}} z^\alpha_{y,t} = 0\,. \tag{3.5}$$

From equation (3.1) we have

$$\lim_\alpha \sum_{\{(y,t)\in S\times S: ryt=\theta\}} z^\alpha_{y,t} = 1\,. \tag{3.6}$$

Now combining equations (3.5) and (3.6) gives

$$\lim_{\alpha} \sum_{(y,t)\in Z(r,w,\theta)} z^{\alpha}_{y,t} = 1\,, \tag{3.7}$$

where $Z(r,w,\theta) = \{(y,t) \in S \times S : t \in Sw,\ ryt = \theta\}$. In particular this latter set is non-empty. □

**Theorem 3.2.** *Let $S$ be a semigroup. Suppose that the Banach algebra $\ell^1(S)$ is biflat. Then there is a constant $C > 0$ such that the following property holds: for each $u \in S$, $N \in \mathbb{N}$ and elements $(r_1, v_1, w_1), \ldots, (r_N, v_N, w_N) \in S \times S \times S$ such that:*

(i) $r_i u = v_i w_i$ $(i = 1, \ldots, N)$*; and,*

(ii) *the sets $Sw_1 \cap [r_1^{-1}(r_1 u)], \ldots, Sw_N \cap [r_N^{-1}(r_N u)]$ are pairwise disjoint.*

*Then necessarily $N \leq C$.*

*Proof.* Set $A = \ell^1(S)$. Since $A$ is biflat, there exists an $A$-bimodule morphism $\rho : A \to (A \widehat{\otimes} A)''$ with $\pi \circ \rho = i_A$. Set $C = \|\rho\|$. For each $i \in \{1, \ldots, N\}$ we set $\theta_i = r_i u = v_i w_i$. Consider the set $Z(r_i, w_i, \theta_i)$ as defined in Lemma 3.1. If $(y,t) \in Z(w_i, u_i, \theta_i)$, then $yt \in Sw_i \cap [r_i^{-1}\theta_i]$. It follows that the sets $Z(w_i, u_i, \theta_i)$ are pairwise disjoint. Now summing equation (3.3) over the sets $Z_i = Z(r_i, w_i, \theta_i)$ gives

$$N = \lim_{\alpha} \sum_{i=1}^{N} \sum_{(y,t)\in Z_i} z^{\alpha}_{y,t} \leq \sup_{\alpha} \sum_{i=1}^{N} \sum_{(y,t)\in Z_i} \left| z^{\alpha}_{y,t} \right| \leq \|\rho(\delta_u)\|_{\pi} = \|\rho\| = C\,.$$

The result follows. □

*Remark* 3.3. The same theorem holds (and with the same constant) if the sets $Sw_i \cap [r_i^{-1}(r_i u)]$ are replaced by $r_i S \cap [(r_i u) w_i^{-1}]$. We obtain this by restricting the sum in equation (3.4) to $s \in (S \setminus Sr) \cap [\theta t^{-1}]$ and then following through the same argument.

We now define two relations on the set $E(S)$. We set

$$u\, R\, v \ \text{ if } \ u \in Sv \cap [v^{-1}v], \quad \text{and} \quad u\, \widetilde{R}\, v \ \text{ if } \ u \in vS \cap [vv^{-1}]\,.$$

Clearly $u\, R\, u$ and $u\, \widetilde{R}\, u$. Suppose that $u\, R\, v$. Then $u = sv$ for some $s \in S$ and $vu = v$. Hence $v \in Su$ and $uv = svv = sv = u$. We see that $v\, R\, u$, and so $R$ is symmetric. Similarly $\widetilde{R}$ is symmetric. A straightforward check shows that $R$ and $\widetilde{R}$ are transitive. Thus $R$ and $\widetilde{R}$ are equivalence relations on $E(S)$. The $R$-class containing $u$ is $Su \cap [u^{-1}u]$ and the $\widetilde{R}$-class containing $u$ is $uS \cap [uu^{-1}]$.

**Theorem 3.4.** *Let $S$ be a semigroup such that $\ell^1(S)$ is biflat. Then $E(S)$ is uniformly locally finite.*

*Proof.* Let $C$ be the constant given in Theorem 3.2. Fix $u \in E(S)$. The collection $\mathcal{A} = \{St \cap [t^{-1}t] : t \in (u]\}$ is a partition of $(u]$. By Theorem 3.2, $|\mathcal{A}| \leq C$. Each $R$-class is a left-zero semigroup and each $\widetilde{R}$-class is a right-zero semigroup. It follows that if $x \neq y$ lie in the same $R$-class then $x$ and $y$ lie in different $\widetilde{R}$-classes. Assume that one of the classes in $\mathcal{A}$ contains $C+1$ or more elements. Then we obtain a partition of $(u]$ by at least $C+1$ $\widetilde{R}$-classes. This contradicts (the remark following) Theorem 3.2. Hence each class in $\mathcal{A}$ contains at most $C$ elements and $|(u]| < C^2$. This holds for each $u \in E(S)$ and so $E(S)$ is uniformly locally finite. □

*Remark* 3.5. The above theorem shows that if $\ell^1(S)$ is $C$-biflat, then $S$ is locally $C^2$-finite. We can improve this estimate for semigroups with commuting idempotents. Indeed, let $S$ be a semigroup such that the idempotents commute. Then $Su \cap [u^{-1}u] = uS \cap [uu^{-1}] = \{u\}$ $(u \in E(S))$. Hence the argument above shows that if $\ell^1(S)$ is $C$-biflat, then $S$ is locally $C$-finite. This agrees with the result in [2] for Clifford semigroups.

A similar argument gives a quantitative version of [6, Theorem 2]. We sketch the details.

**Theorem 3.6.** *Let $S$ be a semigroup such that $\ell^1(S)$ is amenable. Then*

$$|E(S)|^{1/2} \leq AM(\ell^1(S)).$$

*If the idempotents of $S$ commute, then*

$$|E(S)| \leq AM(\ell^1(S)).$$

*Proof.* Let $\mathcal{A}$ and $\widetilde{\mathcal{A}}$ denote the sets of $R$-classes and $\widetilde{R}$-classes respectively. The proof of [6, Theorem 2] shows that $|\mathcal{A}| \leq AM(\ell^1(S))$ and $\left|\widetilde{\mathcal{A}}\right| \leq AM(\ell^1(S))$. Now the result follows using the same arguments as above. □

A better estimate is known for Clifford semigroups. Indeed, let $S$ be a Clifford semigroup, then it follows from [8, Theorem 2.2 and Corollary 1.8] that

$$2\,|E(S)| - 1 \leq AM(\ell^1(E(S))) \leq AM(\ell^1(S))\,.$$

The following theorem is an extension of [2, Theorem 6.1]. The proof in [2] that for a Clifford semigroup $S$, $\ell^1(S)$ biflat implies $S$ is uniformly locally finite uses the fact that $\ell^1(E(S))$ is a retract of $\ell^1(S)$. This is not true for a general inverse semigroup, and so we have to use Theorem 3.4.

**Theorem 3.7.** *Let $S$ be an inverse semigroup. Then:*

(i) *$\ell^1(S)$ is biflat if and only if $S$ is uniformly locally finite and $G_p$ is amenable for each $p \in E(S)$;*

(ii) *$\ell^1(S)$ is biprojective if and only if $S$ is uniformly locally finite and $G_p$ is finite for each $p \in E(S)$.*

*Proof.* (i) Suppose that $\ell^1(S)$ is biflat. Then, by Theorem 3.4 and Proposition 2.14, $S$ is uniformly locally finite. By Theorem 2.18 there is an isomorphism of Banach algebras

$$\ell^1(S) \cong \ell^1\text{-}\bigoplus\{\mathbb{M}_{E(D_\lambda)}(\ell^1(G_{p_\lambda})) : \lambda \in \Lambda\}\,,$$

where $\{D_\lambda : \lambda \in \Lambda\}$ are the $\mathcal{D}$-classes of $S$ and $p_\lambda \in E(D_\lambda)$ $(\lambda \in \Lambda)$. By Proposition 2.3 and Proposition 2.7 each Banach algebra $\ell^1(G_{p_\lambda})$ $(\lambda \in \Lambda)$ is biflat. Hence by Johnson's theorem each group $G_{p_\lambda}$ $(\lambda \in \Lambda)$ is amenable. Now for any $p \in E(S)$ there exists $\lambda$ such that $p\,\mathcal{D}\,p_\lambda$. Then $G_p \cong G_{p_\lambda}$ and hence $G_p$ is amenable.

Conversely, suppose that the conditions on $S$ hold. By Johnson's theorem each Banach algebra $\ell^1(G_{p_\lambda})$ $(\lambda \in \Lambda)$ is 1-biflat. By Proposition 2.7, each Banach algebra $\mathbb{M}_{E(D_\lambda)}(\ell^1(G_{p_\lambda}))$ $(\lambda \in \Lambda)$ is 1-biflat. By Proposition 2.3 and Theorem 2.18, $\ell^1(S)$ is biflat.

(ii) This follows in the same way except we use the fact that $\ell^1(G)$ is biprojective if and only if $G$ is finite, in which case $\ell^1(G)$ is 1-biprojective [3, 3.3.32]. □

## 3.1 Regularity of $S$

In [1] the weak amenability of $\ell^1(S)$ is studied and some necessary conditions for weak amenability are obtained. These are conditions concerning the number of singular elements of $S$. Since biflatness implies weak amenability the results in [1] apply directly to semigroups such that $\ell^1(S)$ is biflat. However, we can use the stronger property of biflatness to obtain stronger conditions with simpler proofs.

**Lemma 3.8.** *Let $S$ be a semigroup such that $\ell^1(S)$ is biflat. Let $t \in S$ with $[tt^{-1}] \neq \emptyset$ and $[t^{-1}t] \neq \emptyset$. Then $t$ is regular.*

*Proof.* There are elements $u, v \in S$ with $tu = vt = t$. Lemma 3.1 shows that the set $Z(t,t,t) \neq \emptyset$, and hence $St \cap [t^{-1}t] \neq \emptyset$. This is equivalent to $t$ being regular. □

**Proposition 3.9.** *Let $S$ be a semigroup such that $\ell^1(S)$ is biflat. Suppose that:*

(i) *$R(S)$ is a subsemigroup of $S$;*

(ii) *$tS = St$ $(t \in N(S))$;*

(iii) *$N(S)$ is finite.*

*Then $S$ is regular.*

*Proof.* Assume towards a contradiction that $N(S) \neq \emptyset$ and take $s_1 \in N(S)$. A biflat Banach algebra is essential and so $S = S^2$. By (i) either $s_1 \in S \cdot N(S)$ or $s_1 \in N(S) \cdot S$. By (ii) this means that there exists $s_2 \in N(S)$ and elements $u_1, v_1 \in S$ with $s_1 = s_2 u_1 = v_1 s_2$.

Continuing like this we obtain a sequence $\{s_1, s_2, \ldots\} \subset N(S)$ such that for each $n \in \mathbb{N}$ there are elements $u_n, v_n \in S$ with

$$s_n = s_{n+1} u_n = v_n s_{n+1} \,.$$

By induction for each $1 \leq k < n$ we have

$$s_k = s_n u_{n-1} \cdots u_k = v_k \cdots v_{n-1} s_n \,.$$

Fix $n \in \mathbb{N}$. Assume towards a contradiction that $s_n \in \{s_1, \ldots, s_{n-1}\}$ so that $s_n = s_k$ for some $k \in \{1, \ldots, n-1\}$. Then we have

$$s_k = s_k u_{n-1} \cdots u_k = v_k \cdots v_{n-1} s_k \,.$$

Hence $[s_k s_k^{-1}] \neq \emptyset$ and $[s_k^{-1} s_k] \neq \emptyset$. By Lemma 3.8, $s_k$ is regular. This is a contradiction and so the elements $\{s_1, \ldots, s_n\}$ are all distinct. This is true for each $n \in N$, and so $N(S)$ is infinite, contradicting (iii). Therefore $N(S) = \emptyset$ and $S$ is regular. □

**Corollary 3.10.** *Let $S$ be a commutative semigroup such that $\ell^1(S)$ is biflat. Suppose that $N(S)$ is finite. Then $S$ is regular.* □

*Remark* 3.11. After this work was completed we learned that Grønbæk and Habibian ([10]) have obtained similar results to ours. The authors in [10] give a different proof of our main theorem for commutative semigroups and use this to characterize the biflatness of $\ell^1(S)$ for a commutative semigroup $S$. Their article also studies the biflatness of Banach algebras graded over semilattices.

**Acknowledgements.** This work forms part of the authors PhD thesis at the University of Leeds, which was supported by an EPSRC grant. The author wishes to thank his supervisor Professor H. G. Dales his advice. Particular thanks are due to Y. Choi for his comments on a draft of this article.

# References

[1] T. D. Blackmore, *Weak amenability of discrete semigroup algebras*, Semigroup Forum **55** (1997), 196–205.

[2] Y. Choi, *Biflatness of $\ell^1$-semilattice algebras*, Semigroup Forum **75** (2007), 253–271.

[3] H. G. Dales, *Banach algebras and automatic continuity*, London Mathematical Society Monographs **24** (Clarendon Press, Oxford), 2000.

[4] H. G. Dales, A. T.-M. Lau, and D. Strauss, *Banach algebras on semigroups and on their compactifications*, Memoirs of American Math. Soc., to appear.

[5] J. Duncan and I. Namioka, *Amenability of inverse semigroups and their semigroup algebras*, Proc. Royal Soc. Edinburgh, Section A, **80** (1978), 309–321.

[6] J. Duncan and A. L. T. Paterson, *Amenability for discrete convolution semigroup algebras*, Mathematica Scandinavica, **66** (1990), 141–146.

[7] G. H. Esslamzadeh, *Banach algebra structure and amenability of a class of matrix algebras with applications*, J. Functional Analysis **161** (1996), 364–383.

[8] M. Ghandehari, H. Hatami, and N. Spronk, *Amenability constants for semilattice algebras*, preprint, see arXiv:0705.4279.

[9] N. Grønbæk, *Amenability of weighted discrete convolution algebras on cancellative semigroups*, Proc. Royal Soc. Edinburgh, Section A **110** (1988), 351–360.

[10] N. Grønbæk and F. Habibian, *Biflatness and biprojectivity of Banach algebras graded over a semilattice*, preprint.

[11] A. Ya. Helemskii, *The homology of Banach and topological algebras*, Kluwer Academic Publishers, Dordrecht, 1986.

[12] J. M. Howie, *Fundamentals of semigroup theory*, London Mathematical Society Monographs, Volume 12 (Clarendon Press, Oxford), 1995.

[13] B. E. Johnson, *Symmetric amenability and the nonexistence of Lie and Jordan derivations*, Math. Proc. Cambridge Philosophical Soc. **120** (1996), 455–473.

[14] B. Steinberg, *Möbius functions and semigroup representation theory*, J. Combinatorial Theory, Series A **113(5)** (2006), 866–881.

[15] ———, *Möbius functions and semigroup representation theory II: character formulas and multiplicities*, Advances in Math. **217** (2008), 1521–1557.

[16] M. C. White, *Injective modules for uniform algebras*, Proc. London Math. Soc. **(3) 73** (1996), 155–184.

Paul Ramsden
Department of Pure Mathematics
University of Leeds
Leeds, LS2 9JT
United Kingdom
pramsden@maths.leeds.ac.uk