Homological properties of semigroup algebras

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Submitted in accordance with the requirements
for the degree of Doctor of Philosophy

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May 2009

Abstract

In this thesis we study projective and injective modules over Banach algebras. We concentrate on classes of Banach algebras generated by groups and semigroups.

Chapter 1 details the necessary background material on Banach algebras and semigroups that we need. We define quantitative versions of projectivity and injectivity, which are important in the Banach algebra setting.

In Chapter 2 we prove a variety of general, intrinsic properties of projective and injective modules over a Banach algebra. The results in this chapter form a useful tool kit for checking projectivity and injectivity in specific cases.

In Chapter 3 we study modules over the measure algebra $M(G)$ for a locally compact group G. We extend results of Dales and Polyakov in [8].

Chapter 4 is some what of an interlude in the main theme of the thesis. In this chapter we introduce and study a structure called a *type-p multi-normed space*, for any $1 \leq p \leq \infty$. This generalizes a construction of Dales and Polyakov [7] and is related to the operator sequence spaces introduced in [26]. We prove various general results and introduce specific examples relating to the group algebra.

Having built up the general machinery of multi-normed spaces we now, in Chapter 5, show how these objects can be used to answer questions about the group algebra. Let G be a locally compact group, and let $1 < p < \infty$. We prove that if $L^p(G)$ is an injective left module over $L^1(G)$, then G is *super p-amenable*, a property very close to that of amenability. In the case where G is discrete we prove that $\ell^p(G)$ is injective if and only if G is super p-amenable.

In Chapter 6 we turn our attention to discrete semigroup algebras. Here we study the biflatness of $\ell^1(S)$ for a semigroup S. We prove that if $\ell^1(S)$ is biflat, then the canonical partial ordering on the idempotents must be uniformly locally finite. We use this to characterize the biflatness of $\ell^1(S)$ for an inverse semigroup S. This generalizes work of Y. Choi in [2]. We also obtain some results about one-sided projectivity of $\ell^1(S)$.

Now in Chapter 7 we investigate the injectivity of $\ell^1(S)$ as a Banach right module over $\ell^1(S)$. For weakly cancellative S this is the same as studying the flatness of the predual module $c_0(S)$. We prove that for many semigroups S such that the Banach algebra $\ell^1(S)$ is non-amenable, $\ell^1(S)$ is not injective. In particular, $\ell^1(S)$ is not injective if S is the bicyclic semigroup or an infinite free semilattice.

Acknowledgements

Firstly, I would like to thank my supervisor Professor H. Garth Dales for all the time and effort that he has given to my mathematics over the last three years.

I would like to thank Yemon Choi (Laval), Zina Lykova (Newcastle) and Alexei Pirkovskii (Moscow) for their comments on various parts of this thesis, for pointing out some errors and suggesting improvements.

This work was supported by an EPSRC grant, which I acknowledge with thanks.

The beautifully typeset pages of this document were created with LaTeX tools and packages provided by the MiKTeX project. All the commutative diagrams were produced using the XY-pic package.

Contents

Chapter 1

Preliminaries

The purpose of this chapter is to sketch the background material on Banach spaces, Banach algebras, and semigroups that we need and to establish notation that will be fixed throughout the thesis. All of the standard results from functional analysis that we use can be found in [31]. For the measure theory background to much of this see [3]. For a more detailed and in depth treatment of Banach spaces see [28]. Recommended references for material on Banach algebras are [4], [18] and [19].

We use the following standard notation for some commonly used sets of numbers; $\mathbb{C}$ denotes the complex numbers: $\mathbb{R}$ denotes the real numbers; $\mathbb{Z}$ denotes the integers; $\mathbb{N} = \{1, 2, \ldots\}$; $\mathbb{N}_n = \{1, 2, \ldots, n\}$; $\mathbb{R}^+ = \{t \in \mathbb{R} : t \geq 0\}$; $\overline{\mathbb{D}} = \{z \in \mathbb{C} : |z| \leq 1\}$; $\mathbb{T} = \{z \in \mathbb{C} : |z| = 1\}$. The *indicator function* of a subset T of a set S is denoted by χ_T, so that $\chi_T(s) = 1 \ (s \in T)$ and $\chi_T(s) = 0 \ (s \in S \setminus T)$. We set $\delta_s = \chi_{\{s\}} \ (s \in S)$. Let $f : S \to E$ be a function from a set S to a vector space E. For $T \subset S$ we define $\chi_T f : S \to E$ by $(\chi_T f)(s) = f(s) \ (s \in T)$ and $(\chi_T f)(s) = 0 \ (s \in S \setminus T)$.

1.1 Banach spaces

The *convex hull* of a non-empty subset S of a linear space E is denoted by $\langle S \rangle$; the set S is *absorbing* if $\bigcup\{tS : t > 0\} = E$; *balanced* if $\alpha S \subset S \ (\alpha \in \overline{\mathbb{D}})$, and *absolutely convex* if S is convex and balanced.

Let K be an absolutely convex, absorbing subset of E. Then the *Minkowski functional* p_K of K, defined by

$$p_K(x) = \inf\{t > 0 : x \in tK\} \quad (x \in E)\,,$$

is a seminorm on E; p_k is a norm if and only if $\bigcap \left\{ \frac{1}{n} K : n \in \mathbb{N} \right\} = \{0\}$.

For a normed space E and $C > 0$, we set $E_{[C]} = \{x \in E : \|x\| \leq C\}$. The set $E_{[1]}$ is the *closed unit ball* of E. Of course $E_{[1]}$ is absolutely convex and absorbing, and $\|x\| = p_{E_{[1]}}(x) \ (x \in E)$.

Let E and F be vector spaces. Then $\mathcal{L}(E,F)$ is the vector space of all linear operators from E to F. Now let E and F be normed spaces. Then $\mathcal{B}(E,F)$ is the vector space of all bounded linear operators from E to F. We write $\mathcal{B}(E)$ for $\mathcal{B}(E,E)$. The identity operator on E is denoted by I_E. The space $\mathcal{B}(E,F)$ becomes a normed space with the *operator norm* $\|T\| = \sup \left\{\|Tx\| : x \in E_{[1]}\right\}$ $(T \in \mathcal{B}(E,F))$, and is complete if F is complete.

Let F be a closed subspace of a Banach space E. A *projection from E onto F* is an element $P \in \mathcal{B}(E)$ with $P^2 = P$ and $P(E) = F$; if there is such a projection with $\|P\| \le C$, then F is *C-complemented* in E. The subspace F is *complemented* if F is C-complemented for some $C \ge 1$. Let E and F be Banach spaces, and let $T \in \mathcal{B}(E,F)$. Then T is *admissible* if $\ker T$, the kernel of T, is complemented in E and $\operatorname{im} T$, the image of T, is closed and complemented in F. Equivalently, T is admissible if there exists $S \in \mathcal{B}(F,E)$ with $T \circ S \circ T = T$.

Dual spaces and weak topologies

Let E be a Banach space. The *dual* of E is the space of continuous linear functionals, $E' = \mathcal{B}(E,\mathbb{C})$. We shall write $\langle x, \lambda\rangle = \lambda(x)$ $(x \in E,\ \lambda \in E')$. We can continue to define higher duals $E'' = (E')', E''' = (E'')', \dots$, etc. The canonical embedding of E into E'' is denoted by ι_E or ι, so that

$$\langle \lambda, \iota_E(x)\rangle = \langle x, \lambda\rangle \quad (x \in E,\ \lambda \in E')\,.$$

The map ι_E is an isometry and the space E is *reflexive* if $\iota(E) = E''$.

Let E and F be Banach spaces, and let $T \in \mathcal{B}(E,F)$. Then the *dual* of T is the map $T' : F' \to E'$ given by

$$\langle x, T'(\lambda)\rangle = \langle T(x), \lambda\rangle \quad (x \in E,\ \lambda \in F')\,.$$

Then $T' \in \mathcal{B}(F', E')$ and $\|T'\| = \|T\|$.

The following theorem is a corollary to the Hahn–Banach separation theorem.

Theorem 1.1.1 ([31, Theorem 3.7]). *Let E be a Banach space. Let $B \subset E$ be a convex, balanced, closed subset, and let $x_0 \in E \setminus B$. Then there exists $\lambda \in E'$ with $|\langle x, \lambda\rangle| \le 1$ $(x \in B)$ and $\langle x_0, \lambda\rangle > 1$.* □

For a subspace F of a normed space E we set

$$F^{\circ} = \{\lambda \in E' : \langle x, \lambda\rangle = 0 \ (x \in F)\}\,.$$

We set $F^{\circ\circ} = (F^{\circ})^{\circ} \subset E''$. The following is a consequence of the Hahn–Banach theorem.

Theorem 1.1.2. *Let F be a closed subspace of a normed space E.*

(i) *For each $\lambda \in F'$, take $\Lambda \in E'$ with $\|\Lambda\| = \|\lambda\|$ and $\Lambda|F = \lambda$. Then the map $\lambda \mapsto \Lambda + F^0$, $F' \to E'/F^0$, is an isometric isomorphism.*

(ii) *Let $q : E \to E/F$ be the quotient map. Then the map $q' : (E/F)' \to F^0$ is an isometric isomorphism.* □

The subspace F is *weakly complemented* if F^0 is complemented in E'. Suppose that $P \in \mathcal{B}(E)$ is a projection from E to F. Then $(I_E - P)' \in \mathcal{B}(E')$ is a projection from E' to F^0. Hence every complemented subspace is weakly complemented.

Definition 1.1.3. Let E be a Banach space. The *weak topology* on E, denoted by $\sigma(E, E')$, is the topology generated by the family of seminorms $\{p_\lambda : \lambda \in E'\}$, where

$$p_\lambda(x) = |\langle x, \lambda\rangle| \quad (x \in E)\,.$$

The *weak-$*$ topology* on E', denoted by $\sigma(E', E)$, is the topology generated by the family of seminorms $\{p_{i(x)} : x \in E\}$.

A net $(x_\alpha) \subset E$ converges to $x \in E$ in $\sigma(E, E')$ if and only if

$$\langle x_\alpha, \lambda\rangle \to \langle x, \lambda\rangle \quad (\lambda \in E')\,,$$

and a net $(\lambda_\alpha) \subset E'$ converges to $\lambda \in E'$ in $\sigma(E', E)$ if and only if

$$\langle x, \lambda_\alpha\rangle \to \langle x, \lambda\rangle \quad (x \in E)\,.$$

The following results show the importance of these topologies.

Theorem 1.1.4. *Let E be a Banach space.*

(i) (Banach–Alaoglu) *The unit ball $E'_{[1]}$ is compact in $\sigma(E', E)$. Every bounded net in E' has a $\sigma(E', E)$-accumulation point and a $\sigma(E', E)$-convergent subnet.*

(ii) (Goldstine) *For each $\Phi \in E''$, there exists a net $(x_\alpha) \subset E_{\|\Phi\|}$ such that $i(x_\alpha) \to \Phi$ in $\sigma(E'', E')$.*

(iii) (Mazur) *For each convex set $S \subset E$, the closures of S in $(E, \|\cdot\|)$ and $(E, \sigma(E, E'))$ are the same.*

(iv) (Principle of local reflexivity) *For each finite-dimensional subspace X of E'', each finite-dimensional subspace Y of E', and each $\varepsilon > 0$, there exists $T : X \to E$ with: $T|X \cap E = I_{X\cap E}$; $\|T\|\,\|T^{-1}|T(X)\| < 1 + \varepsilon$; and*

$$\langle T(\Phi), \lambda\rangle = \langle \lambda, \Phi\rangle \quad (\lambda \in Y, \Phi \in X)\,.$$ □

Operator topologies

Definition 1.1.5. Let E and F be Banach spaces. The *strong operator topology* on $\mathcal{B}(E,F)$, denoted by so, is the topology defined by the family of seminorms $\{p_x : x \in E\}$, where

$$p_x(T) = \|T(x)\| \quad (T \in \mathcal{B}(E,F)).$$

The *weak operator topology* on $\mathcal{B}(E,F)$, denoted by wo, is the topology defined by the family of seminorms $\{p_{x,\lambda} : x \in E,\ \lambda \in F'\}$, where

$$p_{x,\lambda}(T) = |\langle T(x), \lambda\rangle| \quad (T \in \mathcal{B}(E,F)).$$

A net (T_α) converges to T in so if and only if

$$\|T_\alpha(x) - T(x)\| \to 0 \quad (x \in E),$$

and a net (T_α) converges to T in wo if and only if

$$|\langle T_\alpha(x) - T(x), \lambda\rangle| \to 0 \quad (x \in E,\ \lambda \in F').$$

For $\lambda \in E'$ and $x \in F$ we define the rank 1 operator $x \otimes \lambda \in \mathcal{L}(E,F)$ by

$$x \otimes \lambda : y \mapsto \langle y, \lambda\rangle\, x, \quad E \to F.$$

Then $x \otimes \lambda \in \mathcal{B}(E,F)$ and $\|x \otimes \lambda\| = \|x\|\,\|\lambda\|$. We denote by $\mathcal{F}(E,F)$ the collection of finite rank operators from E to F, and set $\mathcal{F}(E) = \mathcal{F}(E,E)$.

Definition 1.1.6. A Banach space E has the *approximation property* if, for each compact set $K \subset E$ and each $\varepsilon > 0$, there exists $T \in \mathcal{F}(E)$ with

$$\|T(x) - x\| < \varepsilon \quad (x \in K).$$

Suppose, further, that T can be chosen with $\|T\| \leq 1$. Then E has the *metric approximation property*.

The projective tensor product

For a clear introduction to the theory of tensor products we recommend the book [33]. Let E and F be normed spaces, and let $E \otimes F$ be the algebraic tensor product of E and F. We define the *projective norm* $\|\cdot\|_\pi$ on $E \otimes F$ by

$$\|z\|_\pi = \inf\left\{\sum_{i=1}^n \|x_i\|\,\|y_i\| : z = \sum_{i=1}^n x_i \otimes y_i,\ x_i \in E,\ y_i \in F\right\} \quad (z \in E \otimes F),$$

where the infimum is taken over all finite representations of z. The completion of $(E \otimes F, \|\cdot\|_\pi)$ is the *projective tensor product* of E and F, which we denote by $E \widehat{\otimes} F$.

The key property of $E \widehat{\otimes} F$ is that, for each continuous bilinear map $B : E \times F \to G$, where G is a Banach space, there is a unique continuous linear map $T : E \widehat{\otimes} F \to G$ with $\|T\| = \|B\|$ and

$$T(x \otimes y) = B(x, y) \quad (x \in E,\, y \in F)\,.$$

Let $S : E \to F$ and $T : X \to Y$ be bounded linear operators between Banach spaces. Then there exists a unique linear operator $S \otimes T : E \widehat{\otimes} X \to F \widehat{\otimes} Y$ such that

$$(S \otimes T)(x \otimes y) = S(x) \otimes T(y) \quad (x \in E,\, y \in X)$$

and, further, $\|S \otimes T\| = \|S\|\,\|T\|$.

Let E be a closed subspace of a Banach space X, and let Y be a Banach space. Then the restriction of the projective norm on $X \otimes Y$ to the algebraic subspace $E \otimes Y$ need not be equal, or even equivalent to the projective norm on $E \widehat{\otimes} Y$. However the following is true.

Proposition 1.1.7 ([33, Proposition 2.4])**.** *Let X and Y be Banach spaces. Let E be a C_1-complemented subspace of X, and let F be a C_2-complemented subspace of Y. Then $E \widehat{\otimes} F$ is a C_1C_2-complemented subspace of $X \widehat{\otimes} Y$ and*

$$\|z\|_{X \widehat{\otimes} Y} \le \|z\|_{E \widehat{\otimes} F} \le C_1C_2 \,\|z\|_{X \widehat{\otimes} Y} \quad (z \in E \widehat{\otimes} F)\,. \qquad \square$$

Another difficultly is to identify the zero vector in the space $E \widehat{\otimes} F$. If either E or F has the approximation property then we can use the following proposition.

Proposition 1.1.8 ([33, Proposition 4.6])**.** *Let E be a Banach space. Then the following are equivalent:*

(i) *E has the approximation property;*

(ii) *for every Banach space F and every $z \in E \widehat{\otimes} F \setminus \{0\}$, there exists $\lambda \in E'$ with $\langle z, \lambda \otimes I_F \rangle \ne 0$;*

(iii) *for every Banach space F and every $z \in E \widehat{\otimes} F \setminus \{0\}$, there exists $\mu \in F'$ with $\langle z, I_E \otimes \mu \rangle \ne 0$.* $\square$

Let E and F be Banach spaces. The dual space $(E \widehat{\otimes} F)'$ is identified with $\mathcal{B}(E, F')$: if $\lambda \in (E \widehat{\otimes} F)'$, then $T_\lambda \in \mathcal{B}(E, F')$ is defined by

$$\langle y,\, T_\lambda(x) \rangle = \langle x \otimes y,\, \lambda \rangle \quad (x \in E,\, y \in F)\,,$$

and the map $\lambda \mapsto T_\lambda$ is an isometric isomorphism.

Some classical Banach spaces

Let X be a topological space, and let $C(X)$ be the set of continuous complex-valued functions on X. Then $C(X)$ is a complex vector space with pointwise operations. We define the *uniform norm* of $f \in C(X)$ by

$$|f|_X = \sup \{|f(x)| : x \in X\} ,$$

and set

$$C^b(X) = \{f \in C(X) : |f|_X < \infty\} .$$

Then $(C^b(X), |\cdot|_X)$ is a Banach space. We define $C_0(X)$ to be the subset of $C(X)$ consisting of functions which *vanish at infinity*: for $f \in C(X)$, $f \in C_0(X)$ if and only if the set $\{x \in X : |f(x)| \geq \varepsilon\}$ is compact for each $\varepsilon > 0$. Then $C_0(X)$ is a closed subspace of $C^b(X)$.

Let $(\Omega, \mathcal{F}, \mu)$ be a measure space, where Ω is a set, $\mathcal{F}$ is a σ-algebra on Ω, and $\mu : \mathcal{F} \to \mathbb{R}^+ \cup \{\infty\}$ is a positive measure. The set of measurable functions $\Omega \to \mathbb{C}$ forms a complex vector space with pointwise operations. Let $p \in [1, \infty)$. We define the *p-norm* of a measurable function $f : \Omega \to \mathbb{C}$ by

$$\|f\|_p = \left(\int_\Omega |f(x)|^p \, \mathrm{d}\mu(x) \right)^{1/p} .$$

The set of measurable functions with finite p-norm is a seminormed space. When we quotient this space by the subspace $\{f : \|f\|_p = 0\}$ we get a Banach space, which we denote by $L^p(\Omega, \mu)$.

A measurable set N is *μ-locally null* if $\mu(N \cap U) = 0$ for all $U \in \mathcal{F}$ with $\mu(U) < \infty$. We define the *essential supremum* of a measurable function $f : \Omega \to \mathbb{C}$ by

$$\|f\|_\infty = \inf \left\{ \sup_{x \in \Omega \setminus N} |f(x)| : N \text{ is } \mu\text{-locally null} \right\} .$$

Again, the set of functions with finite essential supremum is a seminormed space. When we quotient this space by the subspace $\{f : \|f\|_\infty = 0\}$ we get a Banach space, which we denote by $L^\infty(\Omega, \mu)$.

Let $p \in (1, \infty)$. The *conjugate* to p is $q \in (1, \infty)$ which satisfies $1/p + 1/q = 1$; also the conjugates to $p = 1$ and $p = \infty$ are $q = \infty$ and $q = 1$, respectively. Suppose that $p \in (1, \infty)$, or $p = 1$ and μ is σ-finite. Then the dual space of $L^p(\Omega, \mu)$ is identified with $L^q(\Omega, \mu)$, where the duality is specified by

$$\langle f, g \rangle = \int_\Omega f(x) g(x) \, \mathrm{d}\mu(x) \quad (f \in L^p(\Omega, \mu), \, g \in L^q(\Omega, \mu)) .$$

The identification $L^1(\Omega, \mu)' = L^\infty(\Omega, \mu)$ is not true for all measure spaces. However the sufficient condition that μ be σ-finite is not necessary either.

Let Ω be a locally compact topological space. The Borel σ-algebra on Ω is denoted by $\mathcal{B}_\Omega$. Let $M(\Omega)$ be the set of all complex valued, regular Borel measures on $(\Omega, \mathcal{B}_\Omega)$. Then $M(\Omega)$ is a complex vector space with pointwise operations and a Banach space with the *total variation norm* $\|\cdot\|$ given by

$$\|\mu\| = \sup\left\{\sum_{i=1}^{\infty} |\mu(U_i)| : \{U_i : i \in \mathbb{N}\} \text{ is a measurable partition of } \Omega\right\}$$

for $\mu \in M(\Omega)$. We write δ_x for the point mass at $x \in \Omega$. By using the Riesz representation theorem we can identify $M(\Omega)$ with $C_0(\Omega)'$, the duality being specified by

$$\langle f, \mu\rangle = \int_\Omega f(x)\,\mathrm{d}\mu(x) \quad (f \in C_0(\Omega),\ \mu \in M(\Omega))\,.$$

Let S be a set. We denote by $P(S)$ the *power set* of S. Trivially this is a topology on S called the *discrete topology*. Let $p \in [1, \infty]$. We set $\ell^p(S) = L^p(S, P(S), \mu)$ where μ is *counting measure* on S given by $\mu(U) = |U| \ \ (U \in P(S))$. Similarly we define $c_0(S) = C_0((S, P(S)))$. Of course $\ell^p(S)$ and $c_0(S)$ depend only on the cardinality of S. We set $\ell^p = \ell^p(\mathbb{N})$, and for $n \in \mathbb{N}$ we set $\ell^p_n = \ell^p(\mathbb{N}_n)$.

These constructions generalize to vector-valued sequences. Let $\{E_\lambda : \lambda \in \Lambda\}$ be a family of Banach spaces, and let $p \in [1, \infty)$. Then the ℓ^p*-direct sum* of the family $\{E_\lambda : \lambda \in \Lambda\}$ is the Banach space

$$\ell^p\text{-}\bigoplus\{E_\lambda : \lambda \in \Lambda\} = \left\{(x_\lambda) \in \prod_{\lambda\in\Lambda} E_\lambda : \|(x_\lambda)\| = \left(\sum_{\lambda\in\Lambda} \|x_\lambda\|^p\right)^{1/p} < \infty\right\}.$$

The ℓ^∞*-direct sum* is defined similarly. In the case where $E_\lambda = E \ \ (\lambda \in \Lambda)$ we write the ℓ^p-direct sum as $\ell^p(\Lambda, E)$.

Remark 1.1.9. The spaces $C(X)$, $L^p(\Omega, \mu)$ $(1 \le p \le \infty)$, and their duals, all have the metric approximation property; see [33, Examples 4.2 and 4.5].

Integration in Banach spaces

Let (Ω, μ) be a measure space, and let E be a Banach space. A function $f : \Omega \to E$ is *Bochner integrable* if f is μ-measurable and

$$\|f\| = \int_\Omega \|f(t)\|\ \mathrm{d}\mu(t) < \infty\,.$$

The set of Bochner integrable functions is a complete seminormed space. When quotiented out by the subspace $\{f : \|f\| = 0\}$ we get a Banach space, which we denote by $L^1(\Omega, E, \mu)$.

Theorem 1.1.10 ([13, §III, 6.20, Theorem 20])**.** *Let μ be a σ-finite measure on a set Ω, let E and F be Banach spaces, and let $T \in \mathcal{B}(E, F)$. Then, for each $f \in L^1(\Omega, E, \mu)$, $T \circ f \in L^1(\Omega, F, \mu)$ and $\int_\Omega (T \circ f)\,\mathrm{d}\mu = T\left(\int_\Omega f\,\mathrm{d}\mu\right)$.* □

There is a continuous linear operator $J : L^1(\Omega, \mu) \widehat{\otimes} E \to L^1(\Omega, E, \mu)$ given by

$$f \otimes x \mapsto (s \mapsto f(s)x) \quad (f \in L^1(\Omega, \mu),\, x \in E,\, s \in S)\,.$$

By a theorem of Grothendieck J is an isomorphism of Banach spaces (see [10, Example VIII.1.10]).

1.2 Banach algebras and modules

Let A be a Banach algebra, we denote by A-**mod**, by **mod**-A, and by A-**mod**-A the categories of Banach left A-modules, of Banach right A-modules, and of Banach A-bimodules, respectively. Also, if A is unital, we denote by A-**unmod** the category of unital Banach left A-modules, etc. We do not suppose that modules are contractive. For example; if $E \in A$-**mod** then there exists $C > 0$ such that

$$\|a \cdot x\| \leq C \,\|a\|\, \|x\| \quad (a \in A,\, x \in E)\,. \tag{1.1}$$

We shall denote by C_E the minimum constant C that can occur in (1.1). If $C_E \leq 1$, then E is *contractive*.

We shall give definitions only for one category of module; similar definitions apply for modules in other categories.

Let A be a Banach algebra, and let $E \in A$-**mod**. Then the dual space E' has a natural right A-module structure given by

$$\langle x,\, \lambda \cdot a\rangle = \langle a \cdot x,\, \lambda\rangle \quad (a \in A,\, x \in E,\, \lambda \in E')\,.$$

This module is called is the *dual module* of E.

Let A be an algebra, let $S \subset A$ be a subset, and let $E \in A$-**mod**. We set

$$S \cdot E = \{a \cdot x : a \in S,\, x \in E\} \quad \text{and} \quad SE = \operatorname{lin} S \cdot E\,,$$

the linear span of $S \cdot E$. If S is a left ideal of A then SE is a submodule of E. Suppose that A is a Banach algebra and that $E \in A$-**mod**. Then the *essential part* of E is the submodule $\overline{AE}$, and E is *essential* if $\overline{AE} = E$. Now we describe the 'dual' concept. Let $F \in$ **mod**-A. For a subset $S \subset A$ we set

$$F^{\perp S} = \{x \in F : x \cdot S = \{0\}\}\,.$$

If S is a left ideal of A then $F^{\perp S}$ is a closed submodule of F. We set $F^{\perp} = F^{\perp A}$, which is the *annihilator submodule* of F. The A-module F is *faithful* if $F^{\perp} = \{0\}$, and F is an *annihilator module* if $F^{\perp} = F$. Now suppose that $F = E'$ for some $E \in A$-**mod**. Then $F^{\perp S} = (\overline{SE})^0$, and so we have

$$(\overline{SE})' = F/(\overline{SE})^0 = F/F^{\perp S}.$$

Hence, if $\overline{SE} = E$, then $F^{\perp S} = \{0\}$. In particular the dual of an essential module is faithful. Similarly, for $E \in A$-**mod** we set

$$^{S\perp}E = \{x \in E : S \cdot x = \{0\}\}\,.$$

Let A be a Banach algebra, and let $E \in A$-**mod**. Then we write $E = U \oplus_A V$ to indicate that U and V are closed submodules of E with $E = U \oplus V$.

Let A be a Banach algebra, and let $E, F \in A$-**mod**. Then ${}_A\mathcal{B}(E, F)$ is the closed linear subspace of $\mathcal{B}(E, F)$ consisting of the left A-module morphisms. Similarly, we define $\mathcal{B}_A(E, F)$ to be the space of right A-module morphisms when $E, F \in$ **mod**-A. Let $E, F \in A$-**mod**, and let $T \in {}_A\mathcal{B}(E, F)$. Then:

• T is a *retraction* if there exists $S \in {}_A\mathcal{B}(F, E)$ with $T \circ S = I_F$ (so that S is a *right inverse* to T), and in this case F is a *retract* of E;

• T is a *coretraction* if there exists $S \in {}_A\mathcal{B}(F, E)$ with $S \circ T = I_E$ (so that S is a *left inverse* to T).

Let A be a Banach algebra. The *unitization* of A is $A^\flat = A \times \mathbb{C}$ with the product

$$(a, \alpha)(b, \beta) = (ab + \alpha b + \beta a, \alpha\beta) \quad (a, b \in A,\ \alpha, \beta \in \mathbb{C})$$

and the norm

$$\|(a, \alpha)\| = \|a\| + |\alpha| \quad (a \in A,\ \alpha \in \mathbb{C})\,.$$

Thus $A^\flat$ is a unital Banach algebra. Let $E \in A$-**mod**. We define

$$(a, \alpha) \cdot x = a \cdot x + \alpha x \quad (a \in A,\ \alpha \in \mathbb{C},\ x \in E)\,.$$

Then also $E \in A^\flat$-**unmod**. We set $e^\flat = (0, 1)$ and write $(a, \alpha) \in A^\flat$ as $a + \alpha e^\flat$.

Let A be a Banach algebra, and let E be a Banach space. Then $A^\flat \widehat{\otimes} E \in A$-**mod** with module operation specified by

$$a \cdot (b \otimes x) = ab \otimes x \quad (a \in A,\ b \in A^\flat,\ x \in E)\,.$$

Modules of the form $A^\flat \widehat{\otimes} E$ are called *free* modules. Now suppose that $E \in A$-**mod** we define the *canonical morphism* $\pi_E \in {}_A\mathcal{B}(A^\flat \widehat{\otimes} E, E)$ by

$$\pi_E(a \otimes x) = a \cdot x \quad (a \in A^\flat,\ x \in E)\,.$$

Let A be a Banach algebra, and let E be a Banach space. Then $\mathcal{B}(A^\flat, E) \in$ **mod**-A with the module operation

$$(T \cdot a)(b) = T(ab) \quad (a \in A,\ b \in A^\flat,\ T \in \mathcal{B}(A^\flat, E))\,.$$

Modules of the form $\mathcal{B}(A^\flat, E)$ are called *cofree* modules. Now let $E \in$ **mod**-A. We define the *canonical embedding* $\Pi_E : E \to \mathcal{B}(A^\flat, E)$ by the formula

$$\Pi_E(x)(a) = x \cdot a \quad (a \in A^\flat,\ x \in E)\,.$$

Let $E \in A$-**mod**. Then we have an identification of right A-modules and morphisms

$$(A^\flat \widehat{\otimes} E)' = \mathcal{B}(A^\flat, E') \quad \text{and} \quad \pi'_E = \Pi_{E'}\,. \tag{1.2}$$

We shall sometimes write these maps as $\pi_{A^\flat \widehat{\otimes} E}$ and $\Pi_{\mathcal{B}(A^\flat, E)}$ if we wish to emphasize the spaces involved; more often, we shall drop the subscripts completely when this information is clear.

Approximate identities

Definition 1.2.1. Let A be a normed algebra. A *left* [*right*] *approximate identity* for A is a net (e_α) in A such that $\lim_\alpha e_\alpha a = a$ [$\lim_\alpha a e_\alpha = a$]. An *approximate identity* for A is a net (e_α) which is both a left and a right approximate identity.

An approximate identity (of any type) is *bounded* if $\sup_\alpha \|e_\alpha\| < \infty$.

Let A be a Banach algebra with a left approximate identity, and let $E \in A$-**mod**. Then $e_\alpha \cdot x \to x \ \ (x \in E)$ if and only if E is essential. In which case E is also faithful. In particular A is an essential and faithful left A-module.

The following is a corollary to the Cohen factorization theorem.

Theorem 1.2.2 ([4, 2.9.26]). *Let A be a Banach algebra with a bounded left approximate identity, and let $E \in A$-**mod**. Then $\overline{AE} = A \cdot E$ is a weakly complemented submodule of E.* □

Banach ∗-algebras

Definition 1.2.3. Let E be a complex linear space. An *involution* on E is a map $* : E \to E$ such that:

(i) $(\alpha x + \beta y)^* = \overline{\alpha} x^* + \overline{\beta} y^* \ \ (x, y \in E,\ \alpha, \beta \in \mathbb{C})$;

(ii) $(x^*)^* = x \ \ (x \in E)$.

Definition 1.2.4. A *Banach $*$-space* is a Banach space with an isometric involution.

Proposition 1.2.5 ([4, 1.10.4]). *Let E and F be Banach $*$-spaces. Then $F \widehat{\otimes} E$ is a Banach $*$-space for an involution which satisfies*

$$(x \otimes y)^* = x^* \otimes y^* \quad (x \in F,\, y \in E)\,.$$

□

Definition 1.2.6. Let A be a complex algebra. An involution $*$ on A is an *algebra involution* if $(ab)^* = b^* a^* \ \ (a, b \in A)$. A complex algebra with an algebra involution is a *$*$-algebra*. Let A be a $*$-algebra, and let E be an A-bimodule. An involution $*$ on E is a *module involution* if $(a \cdot x)^* = x^* \cdot a^*$ and $(x \cdot a)^* = a^* \cdot x^* \ \ (a \in A, x \in E)$. An A-bimodule with a module involution is a *$*$-module*.

Definition 1.2.7. A *Banach-$*$-algebra* is a Banach algebra with an isometric algebra involution. Let A be a Banach $*$-algebra, and let $E \in A$-**mod**-A. Then E is a *Banach $*$-module* if E has an isometric module involution.

Let A be a Banach $*$-algebra. The formula $(\alpha e^\flat + a)^* = (\overline{\alpha} e^\flat + a^*)$ extends the involution to $A^\flat$. Let $E \in A$-**mod**-A be a $*$-module. Then the dual module E' is also a $*$-module for the involution given by

$$\langle x, \lambda^* \rangle = \overline{\langle x^*, \lambda \rangle} \quad (x \in E,\, \lambda \in E')\,.$$

Let A be a Banach $*$-algebra, and let E be a Banach $*$-space. Then $A^\flat \widehat{\otimes} E$ is a Banach $*$-module.

Some standard constructions

Here we briefly mention some ways of constructing new Banach algebras from old ones.

Let $\{A_\lambda : \lambda \in \Lambda\}$ be a collection of Banach algebras. Then the ℓ^1-direct sum

$$A = \ell^1\text{-}\bigoplus \{A_\lambda : \lambda \in \Lambda\}$$

is a Banach algebra with respect to componentwise multiplication, where multiplication in the λ^{th} component is just multiplication in A_λ. For $a \in A_\lambda$, we let $a\delta_\lambda$ denote the element $(0, \ldots, 0, a, 0, \ldots) \in A$, with a in position λ and 0's elsewhere.

Let A and B be Banach algebras. Then the space $A \widehat{\otimes} B$ becomes a Banach algebra with the multiplication

$$(a_1 \otimes b_1)(a_2 \otimes b_2) = a_1 a_2 \otimes b_1 b_2 \quad (a_1, a_2 \in A,\, b_1, b_2 \in B)\,.$$

Let A be a Banach algebra, and let Λ be a non-empty set. We denote by $\mathbb{M}_\Lambda(A)$ the set of $\Lambda \times \Lambda$ matrices $(a_{ij})_{i,j\in\Lambda}$ with entries in A such that

$$\|(a_{ij})\| = \sum_{i,j} \|a_{ij}\|_A < \infty\,.$$

Then $\mathbb{M}_\Lambda(A)$ is a Banach algebra with usual matrix multiplication. This Banach algebra belongs to the class of *ℓ^1-Munn algebras* introduced in [14]. The *matrix units* in $\mathbb{M}_\Lambda(\mathbb{C})$ are denoted by $E_{i,j}$, so that

$$E_{i,j} E_{k,l} = \delta_{j,k} E_{i,l} \quad (i, j, k, l \in \Lambda)\,,$$

where $\delta_{j,k} = 1$ if $j = k$ and $\delta_{j,k} = 0$ if $j \neq k$.

Proposition 1.2.8. *Let A be a Banach algebra, and let Λ be a non-empty set. Then there is an isometric isomorphism of Banach algebras $\mathbb{M}_\Lambda(A) \cong \mathbb{M}_\Lambda(\mathbb{C}) \widehat{\otimes} A$*

Proof. The map

$$\theta : (a_{i,j}) \mapsto \sum_{i,j} E_{i,j} \otimes a_{i,j}, \quad \mathbb{M}_\Lambda(A) \to \mathbb{M}_\Lambda(\mathbb{C}) \widehat{\otimes} A\,,$$

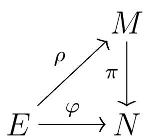
is an isometric algebra isomorphism. □

1.3 Banach homology

We now recall the definitions and basic relationships from Banach homology that the rest of this thesis is concerned with. For full details see [18] and [19]. Quantitative versions of the fundamental notions of projectivity, injectivity and flatness were first explicitly introduced and studied in [40]. This article required all modules to be contractive. We shall modify slightly the definitions given in [40] so that the theory also applies to non-contractive modules. In this thesis we only consider the 'relative' theory, where all lifting and extension problems are admissible.

1.3.1 Projective modules

Let A be a Banach algebra, and let $E \in A$-**mod**. Suppose that we are given a surjective morphism of left A-modules $\pi : M \to N$, and a morphism of left A-modules $\varphi : E \to N$. Consider the following diagram where ρ is simply a set map.



If this diagram commutes, then we say that ρ *lifts* φ. The *lifting problem* (M, N, φ, π) for E is to find a left A-module morphism ρ that lifts φ. The special case where $N = E$ and $\varphi = I_E$ is called the *retraction problem.*

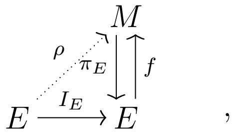
The lifting problem (M, N, φ, π) is *admissible* if π is admissible. We shall write an admissible lifting problem as (M, N, φ, π, f) where $f \in \mathcal{B}(N, M)$ is a right inverse to π. In this case $\rho = f \circ \varphi$ is a bounded linear operator that lifts φ.

Definition 1.3.1. Let A be a Banach algebra, and let $E \in A$-**mod**. Then E is C-*projective* if each admissible lifting problem (M, N, φ, π, f) for E has a solution ρ with $\|\rho\| \le CC_M \|f\| \|\varphi\|$. The module E is *projective* if each admissible lifting problem for E has a solution.

The following proposition characterizes projective modules in terms of a single retraction problem.

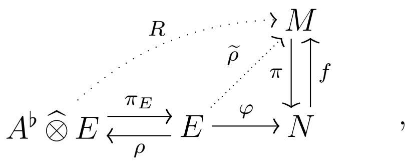
Proposition 1.3.2. *Let A be a Banach algebra, and let $E \in A$-**mod**. Then E is C-projective if and only if the canonical morphism $\pi_E \in {}_A\mathcal{B}(A^\flat \widehat{\otimes} E, E)$ has a right inverse morphism ρ with $\|\rho\| \leq C$.*

Proof. Suppose first that E is C-projective. Set $M = A^\flat \widehat{\otimes} E$ and consider the diagram



where $f \in \mathcal{B}(E, M)$, given by $f : x \mapsto e^\flat \otimes x$ is a right inverse to π_E. By C-projectivity there exists a morphism ρ that is a right inverse to π_E, and such that $\|\rho\| \leq CC_M \|f\| \|I_E\| = C$.

Conversely, suppose that such a map ρ exists. Let (M, N, φ, π, f) be an admissible lifting problem for E. Consider the diagram



where R is given by

$$R(a \otimes x) = a \cdot f \circ \varphi(x) \quad (a \in A^\flat,\, x \in E)\,.$$

Clearly R is a left A-module morphism with $\pi \circ R = \varphi \circ \pi_E$ and $\|R\| \leq C_M \|f\| \|\varphi\|$. We set $\widetilde{\rho} = R \circ \rho$. Then $\pi \circ \widetilde{\rho} = \pi \circ R \circ \rho = \varphi \circ \pi_E \circ \rho = \varphi$ and $\|\widetilde{\rho}\| \leq CC_M \|f\| \|\varphi\|$, as required. □

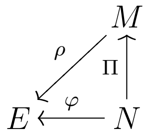
It follows that every projective module is C-projective for some $C > 0$. If the module E is essential, then any morphism $\rho : E \to A^\flat \widehat{\otimes} E$ has range contained in the closed complemented subspace $A \widehat{\otimes} E$. This observation proves the following.

Proposition 1.3.3. *Let A be a Banach algebra, and let $E \in A$-**mod** be essential. Then E is C-projective if and only if the morphism $\pi \in {}_A\mathcal{B}(A \widehat{\otimes} E, E)$ has a right inverse morphism ρ with $\|\rho\| \leq C$.* □

1.3.2 Injective and flat modules

We now consider a problem that is 'dual' to the lifting problem, and is obtained by reversing the direction of the maps. Because of the duality involved it is convenient to give the definitions for right modules.

Let A be a Banach algebra, and let $E \in$ **mod**-A. Suppose that we are given an injective morphism of right A-modules $\Pi : N \to M$ with closed range, and a morphism of right A-modules $\varphi : N \to E$. Consider the following diagram where ρ is simply a set map.



If this diagram commutes, then we say that ρ *extends* φ. The *extension problem* (M, N, φ, Π) for E is to find a right A-module morphism ρ that extends φ. The special case where $N = E$ and $\varphi = I_E$ is called the *coretraction problem.*

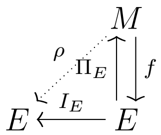
The extension problem (M, N, φ, Π) is *admissible* if Π is admissible. We shall write an admissible extension problem as (M, N, φ, Π, f) where $f \in \mathcal{B}(N, M)$ is a left inverse to π. In this case $\rho = \varphi \circ f$ is a bounded linear operator that extends φ.

Definition 1.3.4. Let A be a Banach algebra, and let $E \in$ **mod**-A. Then E is C-*injective* if each admissible extension problem (M, N, φ, Π, f) for E has a solution ρ with $\|\rho\| \leq CC_M \|f\| \|\varphi\|$. The module E is *injective* if each admissible extension problem for E has a solution.

The following proposition characterizes injective modules in terms of a single coretraction problem.

Proposition 1.3.5. *Let A be a Banach algebra, let $E \in$ **mod**-A, and let $C > 0$. Then E is C-injective if and only if the morphism $\Pi \in \mathcal{B}_A(E, \mathcal{B}(A^\flat, E))$ has a left inverse morphism ρ with $\|\rho\| \leq C$.*

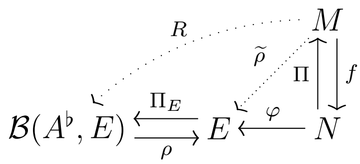
Proof. Suppose first that E is C-injective. Set $M = \mathcal{B}(A^\flat, E)$ and consider the diagram



,

where $f \in \mathcal{B}(M, E)$ is the map $f : T \mapsto T(e^\flat)$, which is a left inverse to Π_E. By C-injectivity there exists a morphism ρ that is a left inverse to Π_E such that $\|\rho\| \leq CC_M \|f\| \|I_E\| = C$.

Conversely, suppose that such a map ρ exists. Let (M, N, φ, Π, f) be an admissible extension problem for E. Consider the diagram



,

where R is given by

$$R(x)(a) = \varphi \circ f(x \cdot a) \quad (a \in A^\flat,\, x \in M)\,.$$

Clearly R is a right A-module morphism with $\|R\| \leq C_M \|f\| \|\varphi\|$, and it is easily checked that $R \circ \Pi = \Pi_E \circ \varphi$. We set $\widetilde{\rho} = \rho \circ R$. Then $\widetilde{\rho} \circ \Pi = \rho \circ R \circ \Pi = \rho \circ \Pi_E \circ \varphi = \varphi$ and $\|\widetilde{\rho}\| \leq CC_M \|f\| \|\varphi\|$, as required. □

It follows that every injective module is C-injective for some $C > 0$. If the module E is faithful, then any morphism $\rho : \mathcal{B}(A^\flat, E) \to E$ satisfies

$$\rho(T \circ P_{\mathbb{C}e^\flat}) = 0 \quad (T \in \mathcal{B}(A^\flat, E)) \,,$$

where $P_{\mathbb{C}e^\flat} \in \mathcal{B}(A^\flat)$ is a projection onto the subspace $\mathbb{C}e^\flat$. From this observation we can prove the following.

Proposition 1.3.6 ([8, Proposition 1.7]). *Let A be a Banach algebra, and let $E \in$ **mod**-A be faithful. Then E is C-injective if and only if the morphism $\Pi : E \to \mathcal{B}(A^\flat, E)$ has a left inverse morphism ρ with $\|\rho\| \leq C$.* □

It follows from (1.2) that the dual E' of a C-projective left A-module E is a C-injective right A-module. However the converse in general is false. This motivates the following definition.

Definition 1.3.7. Let A be a Banach algebra, and let $E \in A$-**mod**. Then E is *C-flat* if E' is C-injective in **mod**-A. The module E is *flat* if E' is injective in **mod**-A.

Thus every projective module is flat. Similar definitions apply in the cases where $E \in$ **mod**-A and where $E \in A$-**mod**-A. There are examples given in [8] to show that the only general relationship between the notions 'E is projective', 'E is injective' and 'E is flat' is the trivial one; E is projective implies that E is flat.

The following is a useful characterization of flat modules.

Proposition 1.3.8 ([18, Exercise VII2.8]). *Let A be a Banach algebra, and let $E \in A$-**mod**. Then E is C-flat if and only if there exists a left A-module morphism $\rho : E \to (A^\flat \widehat{\otimes} E)''$ with $\|\rho\| \leq C$ such that the following diagram commutes*

$$\begin{array}{ccc} E & \xrightarrow{\rho} & (A^\flat \widehat{\otimes} E)'' \\ & {\scriptstyle \iota_E} \searrow & \downarrow {\scriptstyle \pi''} \\ & & E'' \end{array} \quad .$$

Proof. Suppose first that E is C-flat in A-**mod**. Let $\rho \in \mathcal{B}_A(\mathcal{B}(A^\flat, E'), E')_{[C]}$ with $\rho \circ \Pi = I_{E'}$. Set $\widetilde{\rho} = \rho' \circ \iota_E : E \to (A^\flat \widehat{\otimes} E)''$. Then $\widetilde{\rho}$ has the required properties.

Conversely, let ρ be as specified above. Let $\iota : (A^\flat \widehat{\otimes} E)' \to (A^\flat \widehat{\otimes} E)'''$ be the natural embedding. Define $\widetilde{\rho} : (A^\flat \widehat{\otimes} E)' \to E'$ by $\widetilde{\rho}(T) = \rho'(\iota(T))$ $(T \in (A^\flat \widehat{\otimes} E)')$. Then $\widetilde{\rho}$ is a right A-module morphism and for $x \in E$, $\lambda \in E'$ we have

$$\langle x, \widetilde{\rho}(\Pi\lambda)\rangle = \langle \rho(x), \iota(\Pi\lambda)\rangle = \langle \Pi\lambda, \rho(x)\rangle = \langle \lambda, \pi'' \circ \rho(x)\rangle = \langle x, \lambda\rangle \ ,$$

which shows that $\widetilde{\rho} \circ \Pi = I_{E'}$. Therefore E is C-flat in A-**mod**. $\square$

If E is essential, then we can replace $A^\flat \widehat{\otimes} E$ by $A \widehat{\otimes} E$ in the above proposition.

Example 1.3.9. Let A be a Banach algebra with a right identity e. Then the map

$$\rho : a \mapsto a \otimes e, \quad A \to A \widehat{\otimes} A\,,$$

is a left A-module morphism and a right inverse to π_A. Therefore A is a projective left A-module.

Now let A be a Banach algebra with a bounded right approximate identity (e_α). Define $\rho_\alpha : A \to (A \widehat{\otimes} A)''$ by

$$\rho_\alpha(a) = a \otimes e_\alpha \quad (a \in A)\,.$$

Then (ρ_α) is a bounded net in $\mathcal{B}(A, (A \widehat{\otimes} A)'') = \big(A \widehat{\otimes} (A \widehat{\otimes} A)'\big)'$. Let ρ be a weak-$*$ accumulation point of this net. It is easily checked that ρ is a left A-module morphism with $\pi_A'' \circ \rho = \iota_A$. Therefore A is a flat left A-module.

1.3.3 Amenable and biflat Banach algebras

Let A be a Banach algebra. Then the projective tensor product $A \widehat{\otimes} A$ is a Banach A-bimodule, where the multiplication is specified by

$$a \cdot (b \otimes c) = ab \otimes c \quad \text{and} \quad (b \otimes c) \cdot a = b \otimes ca \quad (a, b, c \in A)\,.$$

Definition 1.3.10. Let A be a Banach algebra, and let $C > 0$. Then A is:

- *C-biprojective* if there exists an A-bimodule morphism $\rho : A \to A \widehat{\otimes} A$ with $\pi \circ \rho = I_A$ and $\|\rho\| \le C$;
- *C-biflat* if there exists an A-bimodule morphism $\rho : (A \widehat{\otimes} A)' \to A'$ with $\rho \circ \pi' = I_{A'}$ and $\|\rho\| \le C$;
- *C-contractible* if there exists $d \in A \widehat{\otimes} A$ with

$$a \cdot d - d \cdot a = 0, \quad a\pi(d) = a \quad (a \in A) \quad \text{and} \quad \|d\| \le C\,;$$

- *C-amenable* if there exists a net $(d_\alpha) \subset A \widehat{\otimes} A$ with

$$a \cdot d_\alpha - d_\alpha \cdot a \to 0, \quad a\pi(d_\alpha) \to a \quad (a \in A), \quad \text{and} \quad \sup \|d_\alpha\| \le C\,.$$

For an amenable or biflat Banach algebra A we set

$$AM(A) = \inf \{C > 0 : A \text{ is } C\text{-amenable}\}\,,$$

and

$$BF(A) = \inf \{C > 0 : A \text{ is } C\text{-biflat}\} ,$$

respectively. It is known that a Banach algebra A is amenable if and only if A is biflat and has a bounded approximate identity [4, Theorem 2.9.65], in which case we have the following relationship between the associated constants

$$BF(A) \le AM(A) \le BF(A) \cdot \inf \left\{ \sup_\alpha \|e_\alpha\| : (e_\alpha) \text{ is a bai for } A \right\} .$$

Proposition 1.3.11. *Let A be a Banach algebra, and let $E \in A$-**mod** or $E \in$ **mod**-A.*

(i) *If A is C-contractible, then E is C-projective.*

(ii) *If A is C-amenable, then E is C-flat.* □

There is a generalization of this result which applies to biflat and biprojective Banach algebras, for example if A is biflat, then every module of the form $A \widehat{\otimes}_A E$ ($E \in A$-**mod**) is flat (see [19, VII.1.57]).

1.4 Locally compact groups

Definition 1.4.1. A group G is a *locally compact group* if G is also a locally compact Hausdorff space, and the maps $(s,t) \mapsto st$, $G \times G \to G$, and $s \mapsto s^{-1}$, $G \to G$, are continuous.

The identity of a group is denoted by e_G. A locally compact group is *compact* [*discrete*] if the underlying topological space is compact [discrete]. Each locally compact group carries an analogue of Lebesgue measure on $(\mathbb{R}, +)$. The following theorem is one of the foundations of abstract harmonic analysis.

Theorem 1.4.2. *Let G be a locally compact group. Then there is a positive, regular Borel measure m on G such that:*

(i) *$m(U) > 0$ for each non-empty open subset $U \subset G$;*

(ii) *$m(K) < \infty$ for compact subset $K \subset G$;*

(iii) *$m(sE) = m(E)$ for each $s \in G$ and $E \in \mathcal{B}_G$.* □

The measure given in the above theorem is unique up to multiplication by a positive constant. It is called the (*left*) *Haar measure* of G. For each $s \in G$ the map $E \mapsto m(Es)$, $\mathcal{B}_G \to [0, \infty]$, is a left Haar measure. Hence there exists $\Delta(s) > 0$ such that

$$m(Es) = \Delta(s) m(E) \quad (E \in \mathcal{B}_G) .$$

The function $\Delta : s \mapsto \Delta(s)$, $G \to (\mathbb{R}^+ \setminus \{0\}, \times)$, is a continuous group homomorphism, called the *modular function* of G.

1.5 Group and measure algebras

Now we introduce our main objects of study. Let G be a locally compact group. We define *convolution multiplication* $\star$ on $M(G)$ by

$$(\mu \star \nu)(E) = \int_G \nu(t^{-1}E)\,\mathrm{d}\mu(t) = \int_G \mu(Et^{-1})\,\mathrm{d}\nu(t) \quad (E \in \mathcal{B}_G,\ \mu, \nu \in M(G)).$$

Then $M(G)$ is a unital Banach algebra with identity δ_{e_G}. There is an involution $*$ on $M(G)$ given by

$$\mu^*(E) = \overline{\mu(E^{-1})} \quad (E \in \mathcal{B}_G,\ \mu \in M(G)).$$

With this involution $M(G)$ is a Banach $*$-algebra. For $\mu \in M(G)$ we define $\overline{\mu} \in M(G)$ by $\overline{\mu}(E) = \overline{\mu(E)}$ $(E \in \mathcal{B}_G)$. There is always one character on $M(G)$; this is the *augmentation character* φ_G, defined by

$$\varphi_G(\mu) = \mu(G) \quad (\mu \in M(G)).$$

Let m be left Haar measure on G. The subspace of $M(G)$ consisting of measures which are absolutely continuous with respect to m is denoted by $M_a(G)$. This is a closed ideal with a Banach space complement. By the Radon–Nikodym theorem we can identify $M_a(G)$ with $L^1(G) = L^1(G, m)$. The convolution product on $L^1(G)$ becomes

$$(f \star g)(s) = \int_G f(t)g(t^{-1}s)\,\mathrm{d}m(t) \quad (f, g \in L^1(G)),$$

where the integral is defined for almost all $s \in G$. The Banach algebra $L^1(G)$ contains an approximate identity bounded by 1 and is unital if and only if G is discrete. The restriction of φ_G to $L^1(G)$ has the form

$$\varphi_G(f) = \int_G f(t)\,\mathrm{d}m(t) \quad (f \in L^1(G)).$$

For details of the above theory of $M(G)$, see [20], especially Theorem (19.20), and [4, §3.3].

1.5.1 Modules over $L^1(G)$ and $M(G)$

Let G be a locally compact group. There are various canonical modules over $L^1(G)$ and $M(G)$. Let $E \in L^1(G)$-**mod** be essential. Then the module multiplication extends to $M(G)$ so that $E \in M(G)$-**mod**. This follows from [4, 2.9.51] and the fact that the multiplier algebra of $L^1(G)$ is identified with $M(G)$ [4, 3.3.40].

Let $E \in M(G)$-**mod**-$M(G)$. Then we define

$$s \cdot x = \delta_s \cdot x, \quad x \cdot s = x \cdot \delta_s \quad (x \in E,\ s \in G).$$

Proposition 1.5.1 ([23, Proposition 2.1])**.** *Let G be a locally compact group, and let $E \in L^1(G)$-**mod** be essential. Then for each $f \in L^1(G)$ and $x \in E$ we have*

$$f \cdot x = \int f(t)t \cdot x \, \mathrm{d}m(t) \, .$$

Proof. We first prove the result in the special case where $E = L^1(G)$. Take $f, g \in L^1(G)$, and $\lambda \in L^\infty(G)$. Then $t \mapsto f(t)t \star g \in L^1(G, L^1(G))$ and we have

$$\begin{aligned} \left\langle \int_G f(t)t \star g \, \mathrm{d}m(t), \lambda \right\rangle &= \int_G f(t) \langle t \star g, \lambda \rangle \, \mathrm{d}m(t) \quad \text{(by Theorem 1.1.10)} \\ &= \int_G f(t) \int_G \lambda(s) g(t^{-1}s) \, \mathrm{d}m(s) \, \mathrm{d}m(t) \\ &= \langle f \star g, \lambda \rangle \quad \text{(by Fubini)} \, . \end{aligned}$$

Now take $x \in E$ and $f \in L^1(G)$. Then $x = g \cdot y$ for some $g \in L^1(G)$ and $y \in E$. We have

$$\begin{aligned} f \cdot x = (f \star g) \cdot y = \Pi(y) \left(\int_G f(t)t \star g \, \mathrm{d}m(t) \right) \\ = \int_G f(t)(t \star g) \cdot y \, \mathrm{d}m(t) \quad \text{(by Theorem 1.1.10)} \\ = \int_G f(t)t \cdot x \, \mathrm{d}m(t) \, , \end{aligned}$$

as required. □

The module $L^p(G)$

Let G be a locally compact group, and take $1 \leq p < \infty$. Set $L^p(G) = L^p(G, m)$. For $f \in L^p(G)$ and $\mu \in M(G)$ we define

$$(\mu \star_p f)(s) = \int_G f(t^{-1}s) \, \mathrm{d}\mu(t), \quad (f \star_p \mu)(s) = \int_G f(st^{-1}) \Delta^{1/p}(t^{-1}) \, \mathrm{d}\mu(t) \, ,$$

where again the integrals are defined for almost all $s \in G$. With these actions $L^p(G)$ is a unital $M(G)$-bimodule and an essential left (and right) $L^1(G)$-module. For $f \in L^p(G)$ we have

$$(t \cdot f)(s) = f(t^{-1}s), \quad (f \cdot t)(s) = f(st^{-1}) \Delta(t^{-1})^{1/p} \quad (s, t \in G) \, .$$

There is an involution on $L^p(G)$ given by

$$f^*(s) = \overline{f(s^{-1})} \Delta^{1/p}(s^{-1}) \quad (s \in G, \; f \in L^p(G)) \, .$$

With this involution $L^p(G)$ is a Banach $L^1(G)$-$*$-module. In the case where $p = 1$ this is just the restriction to $L^1(G)$ of the involution on $M(G)$.

Now suppose that $p \in (1, \infty)$, and let q be the conjugate index to p. Let $\cdot_q$ denote the dual actions on the space $L^q(G) = L^p(G)'$. Then $\cdot_q$ satisfies

$$\mu \cdot_q g = g \star_q \overline{\mu}^*, \quad g \cdot_q \mu = \overline{\mu}^* \star_q g \quad (g \in L^q(G),\, \mu \in M(G))\,.$$

The map

$$g \mapsto \overline{g}^*, \quad (L^q(G), \star_q) \to (L^q(G), \cdot_q)$$

is an isometric isomorphism of $M(G)$-bimodules.

The modules $C_0(G)$ and $L^\infty(G)$

Let G be a locally compact group. Then there exists a σ-compact open and closed subgroup H and a subset $S \subset G$ such that G is a disjoint union

$$G = \bigcup \{sH : s \in S\}\ .$$

We can identify $L^1(G)$ with ℓ^1-$\bigoplus\{L^1(sH) : s \in S\}$. Hence we may identify the dual $M(G)$-module $L^1(G)'$ with ℓ^∞-$\bigoplus\{L^\infty(sH) : s \in S\} = L^\infty(G)$. For $f \in L^1(G)$ and $\lambda \in L^\infty(G)$ the module actions are given by

$$(f \cdot \lambda)(s) = \int_G f(t)\lambda(st)\,\mathrm{d}m(t), \quad (\lambda \cdot f)(s) = \int_G f(t)\lambda(ts)\,\mathrm{d}m(t) \quad (s \in G)\,.$$

For $\lambda \in L^\infty(G)$ we have

$$(t \cdot \lambda)(s) = \lambda(st), \quad (\lambda \cdot t)(s) = \lambda(ts) \quad (s, t \in G)\,.$$

Further, $L^\infty(G)$ is a Banach $L^1(G)$-$*$-module for the involution

$$\lambda^*(s) = \overline{\lambda(s^{-1})} \quad (s \in G)\,.$$

The Banach space $C_0(G)$ is a $*$-closed, $M(G)$-sub-bimodule of $L^\infty(G)$; further $C_0(G)$ is an essential left (and right) $L^1(G)$-module. The identification of Banach spaces $M(G) = C_0(G)'$ is also an identification of $M(G)$-bimodules.

1.5.2 Amenable locally compact groups

In the following definition we consider $L^1(G)''$ as a dual $M(G)$-module.

Definition 1.5.2. Let G be a locally compact group. An element $\Lambda \in L^1(G)''$ is a *mean* on G if $1 = \langle 1, \Lambda\rangle = \|\Lambda\|$, and *left invariant* if $s \cdot \Lambda = \Lambda$ $(s \in G)$; G is *amenable* if there exists a left invariant mean in $L^1(G)''$.

Example 1.5.3. Every compact and every abelian locally compact group is amenable. Non-amenable examples include any locally compact group containing the free group on 2-generators, $\mathbb{F}_2$ as a *closed* subgroup.

Let G be a locally compact group. We set

$$P(G) = \left\{ f \in L^1(G) : f \geq 0,\, \|f\| = 1 \right\} .$$

We shall use the following characterization, known as *Reiter's condition.*

Proposition 1.5.4 ([29, Proposition (0.8)]). *Let G be a locally compact group. Then G is amenable if and only if there is a net $(f_\alpha) \subset P(G)$ such that* $\lim \|s \cdot f_\alpha - f_\alpha\| = 0$ *for each $s \in G$.* □

Theorem 1.5.5 (Johnson). *Let G be a locally compact group. Then the Banach algebra $L^1(G)$ is amenable if and only if the locally compact group G is amenable.* □

1.6 Semigroups

Our standard reference for the algebraic theory of semigroups is [21].

Definition 1.6.1. A *semigroup* is a non-empty set S together with an associative, binary operation

$$S \times S \to S : (s,t) \mapsto st\,.$$

A subset $I \subset S$ is a *left* [*right*] *ideal* if $SI \subset I$ [$IS \subset I$]. A subset which is both a left and a right ideal is an *ideal*.

We do not suppose that a semigroup has an identity. Of course a group is a semigroup, but it is not always helpful to think of semigroups as a generalization of groups. There are vastly more semigroups than groups.

Let S be a semigroup. If S is unital then the identity is denoted by e_S. The forced unitization of S is denoted by $S^\flat$.

The study of ideals generated by elements in a semigroup leads to several important equivalence relations on a semigroup known as *Green's equivalences*. The equivalence relation that will be important to us is the following.

Definition 1.6.2. Let S be a semigroup, and let $s, t \in S$. We define a relation $\mathcal{D}$ on S by $s\,\mathcal{D}\,t$ if and only if there exists $a \in S$ with $S^\flat s = S^\flat a$ and $aS^\flat = tS^\flat$.

For a proof that this is an equivalence relation see [21, Chapter 2].

An element $p \in S$ is *idempotent* if $p^2 = p$. The set of idempotents is denoted by $E(S)$. A semigroup S is a *semilattice* if S is commutative and $E(S) = S$. The *canonical partial order* on $E(S)$ is given by

$$p \leq q \iff p = pq = qp \quad (p, q \in E(S))\,.$$

An idempotent p is *maximal* if $p = q$ whenever $p \leq q$.

Example 1.6.3. Motivating examples for us are $\mathbb{N}_\wedge = (\mathbb{N}, \min)$ and $\mathbb{N}_\vee = (\mathbb{N}, \max)$. The canonical partial order on $\mathbb{N}_\wedge$ agrees with the usual order on $\mathbb{N}$, and on $\mathbb{N}_\vee$ it is the reverse order.

Cancellativity

We shall use the following notation introduced by Grønbæk in [16]. For $s, t \in S$ we define the sets

$$[st^{-1}] = \{u \in S : ut = s\} \ ,$$
$$[t^{-1}s] = \{u \in S : tu = s\} \ .$$

Let S be a semigroup. An element $t \in S$ is *left cancellable* if $u = v$ whenever $tu = tv$. Equivalently we require that $|[t^{-1}s]| \leq 1 \ \ (s \in S)$. *Right cancellable* elements are defined similarly. The semigroup S is *cancellative* if each element is both left and right cancellative. The semigroup S is *weakly left* (respectively, *right*) *cancellative* if $[t^{-1}s]$ (respectively, $[st^{-1}]$) is finite for each $s, t \in S$, and *weakly cancellative* if it is both weakly left cancellative and weakly right cancellative.

Regularity

Let S be a semigroup, and let $s \in S$. An element $s^* \in S$ is an *inverse* of s if

$$ss^*s = s \quad \text{and} \quad s^*ss^* = s^* \,.$$

In general an inverse will not be unique. An element $s \in S$ is *regular* if there exists $t \in S$ with $sts = s$. Clearly if s has an inverse then s is regular; less obviously the converse is also true. Indeed suppose that there is $t \in S$ with $sts = s$. Set $s^* = tst$. Then we have

$$ss^*s = (sts)ts = sts = s \text{ and } s^*ss^* = t(sts)tst = t(sts)t = s^* \,.$$

The set of regular elements of S is denoted by $R(S)$. The semigroup S is *regular* if $S = R(S)$. An element that is not regular is called *singular*, and the set of singular elements is denoted by $N(S)$.

Partially ordered sets

Let P be a partially ordered set. For $p \in P$, we define $(p] = \{x \in P : x \leq p\}$ and $[p) = \{x \in P : p \leq x\}$. Then P is *locally finite* if $(p]$ is finite for each $p \in P$, and P is *locally C-finite* for some constant $C > 1$ if $|(p]| < C$ for each $p \in P$. A partially ordered set that is locally C-finite for some C is *uniformly locally finite*.

Let P be a partially ordered set. Following the notation of [2] we define $\mathrm{Sch} : \ell^1(P) \to \ell^\infty(P)$ by

$$\mathrm{Sch}(\delta_t) = \chi_{(t]} \quad (t \in S)\,.$$

The map Sch is called the *Schützenburger representation* of P; see [2, §4] and the references therein.

Proposition 1.6.4 ([2, Proposition 6.5])**.** *Let P be a uniformly locally finite partially ordered set. Then the range of* Sch *is contained in $\ell^1(P)$ and* $\mathrm{Sch} : \ell^1(P) \to \ell^1(P)$ *is an isomorphism of Banach spaces.*

Proof. Set $C = \sup_{t\in P} |(t]|$. The map Sch is a linear operator with norm

$$\|\mathrm{Sch}\| = \sup_{t\in P} \|\mathrm{Sch}(\delta_t)\|_1 = \sup_{t\in P} |(t]| = C\,.$$

The map Sch has an inverse $\mathrm{Sch}^{-1} : \ell^1(P) \to \ell^1(P)$ given by

$$\mathrm{Sch}^{-1}(\delta_t) = \sum_{s\in(t]} \mu_{(t]}(s,t)\delta_s \quad (t \in P)\,,$$

where $\mu_{(t]}$ is the Möbius *function* for the partially ordered set $(t]$; see [35, §3.7]. By [2, Theorem 5.5] Sch^{-1} is a bounded linear operator with $\left\|\mathrm{Sch}^{-1}\right\| \leq 2^{C-1}$. □

Inverse semigroups

Definition 1.6.5. Let S be a semigroup. Then S is an *inverse semigroup* if S is regular and every element has a unique inverse.

We shall denote the inverse of an element s in an inverse semigroup by s^{-1}. The following proposition gives an important equivalent definition, and is often the easiest way to show that a semigroup is an inverse semigroup.

Proposition 1.6.6 ([21, Proposition 5.1.1])**.** *Let S be a semigroup. Then S is an inverse semigroup if and only if S is regular and the idempotents commute.* □

Let S be an inverse semigroup. There is a natural partial order on S given by $s \leq t \Leftrightarrow s = ss^{-1}t$. This agrees with the usual partial order on $E(S)$.

Remark 1.6.7. There are various equivalent definitions of the partial order on S; see [21, Proposition 5.2.1]. For example $s \leq t \Leftrightarrow s = ts^{-1}s \Leftrightarrow s = tp$ for some $p \in E(S) \Leftrightarrow s = pt$ for some $p \in E(S)$.

Proposition 1.6.8. *Let S be an inverse semigroup. Suppose that $(E(S), \leq)$ is [uniformly] locally finite. Then $(S, \leq)$ is [uniformly] locally finite.*

Proof. For $t \in S$ we set $(t]_S = \{s \in S : s \leq t\}$ and $(t]_E = \{s \in E(S) : s \leq t\}$. For $p \in E(S)$ we have $(p]_S = (p]_E$. Fix $t \in S$. The inclusion $\{tp : p \in (t^{-1}t]_E\} \subset (t]_S$ is clear. Now take $s \leq t$. Then $s = ts^{-1}s = t(t^{-1}ts^{-1}s)$ and $t^{-1}ts^{-1}s \leq t^{-1}t$, hence

$$(t]_S = \{tp : p \in (t^{-1}t]_E\} .$$

The result follows. □

Definition 1.6.9. Let S be an inverse semigroup. Then S is [*locally finite* / C*-locally finite* / *uniformly locally finite*] respectively if the partially ordered set $(E(S), \leq)$ has the corresponding property.

We shall use the following characterization of $\mathcal{D}$-classes in an inverse semigroup.

Proposition 1.6.10 ([21, Proposition 5.1.2(4)])**.** *Let S be an inverse semigroup, and let $s, t \in S$. Then $s\,\mathcal{D}\,t$ if and only if there exists $x \in S$ with $s^{-1}s = xx^{-1}$ and $t^{-1}t = x^{-1}x$.* □

Let S be an inverse semigroup, and let $p \in E(S)$. We set

$$G_p = \{s \in S : ss^{-1} = s^{-1}s = p\} .$$

Then G_p is a group with identity p and G_p contains any other subgroup of S with identity p. Thus G_p is called the *maximal subgroup of S at p*. Suppose that idempotents p and q lie in the same $\mathcal{D}$-class. Take $x \in S$ with $p = xx^{-1}$ and $q = x^{-1}x$. Then the map

$$s \mapsto x^{-1}sx, \quad G_p \to G_q ,$$

is an isomorphism.

Definition 1.6.11. Let S be a semigroup. Then S is a *Clifford semigroup* if S is an inverse semigroup such that $ss^{-1} = s^{-1}s \;\; (s \in S)$.

Let S be a Clifford semigroup, and let $s \in S$. Then $s \in G_{ss^{-1}}$, and hence S is a disjoint union of the groups $G_p \;\; (p \in E(S))$. The next proposition tells us that a Clifford semigroup is a *semilattice of groups*.

Proposition 1.6.12 ([21, Theorem 4.2.1])**.** *Let S be a Clifford semigroup. Then for each $p, q \in E(S)$ we have*

$$G_pG_q \subset G_{pq} .$$

□

1.7 Semigroup algebras

Let S be a semigroup. The *semigroup algebra* $\ell^1(S)$ is the completion in the ℓ^1-norm of the algebra $\mathbb{C}S$. It is the *Banach algebra generated by the semigroup*. The *convolution product* $\star$ on $\ell^1(S)$ is uniquely defined by requiring that

$$\delta_s \star \delta_t = \delta_{st} \quad (s, t \in S) .$$

For $f, g \in \ell^1(S)$ we have

$$(f \star g)(r) = \sum_{st=r} f(s)g(t) \quad (r \in S)$$

where the sum is defined to be zero if $\{(s,t) : st = r\} = \emptyset$. We identify $\ell^1(S)^\flat$ with $\ell^1(S^\flat)$. These Banach algebras have been studied by many authors. A recent memoir is [5], which contains many references to original papers.

Identities

Let S be a semigroup. The semigroup algebra $\ell^1(S)$ may have an identity even if S is non-unital. For example let S be a finite semilattice. Then $\ell^1(S)$ has an identity [15, Proposition 1.1].

Results about identities and approximate identities in semigroup algebras have been proved by several authors. Necessary and sufficient conditions for $\ell^1(S)$ to have a bounded approximate identity for an inverse semigroup S were given in [11]. These were generalized to other semigroups in [16].

Lemma 1.7.1. *Let S be a semigroup, let $a, b \in \ell^1(S)$, and let t be a right cancellable element. Then $a = b$ whenever $a \star \delta_t = b \star \delta_t$.*

Proof. The equation $a \star \delta_t = b \star \delta_t$ can be written

$$\sum_{r \in S} \left(\sum_{s \in [rt^{-1}]} a_s \right) \delta_r = \sum_{r \in S} \left(\sum_{s \in [rt^{-1}]} b_s \right) \delta_r .$$

Hence for each $r \in S$ we have

$$\sum_{s \in [rt^{-1}]} a_s = \sum_{s \in [rt^{-1}]} b_s .$$

Since t is a right cancellable element $[(ut)t^{-1}] = \{u\}$ for each $u \in S$. Hence making the substitution $r = ut$ in the above equation gives $a_u = b_u$ $(u \in S)$. □

Lemma 1.7.2. *Let S be a semigroup such that the Banach algebra $\ell^1(S)$ has a left identity. Suppose that S contains a right cancellable element. Then:*

(i) *S has a left identity e_S;*

(ii) *for each right cancellable element t we have $te_S = t$.*

Proof. (i) Let $e_A = \sum_{s\in S} e_s\delta_s$ be a left identity for $\ell^1(S)$, and let t be a right cancellable element, so that $|[tt^{-1}]| \le 1$. We have

$$\delta_t = \sum_{s\in S} e_s\delta_{st} = \sum_{r\in S}\left(\sum_{s\in[rt^{-1}]} e_s\right)\delta_r\,.$$

Hence $\sum_{s\in[tt^{-1}]} e_s = 1$ and in particular the set $[tt^{-1}]$ is non-empty, say $[tt^{-1}] = \{u\}$. Then

$$\delta_u \star \delta_t = \delta_t = e_A \star \delta_t\,.$$

By Lemma 1.7.1, $\delta_u = e_A$, and so u is a left identity for S.

(ii) Let t be a right cancellable element and e_S a left identity for S. Then $t^2 = t(e_St) = (te_S)t$, which implies that $t = te_S$. □

Module actions

Let S be a semigroup, and let $E \in \ell^1(S)$-**mod**-$\ell^1(S)$. We shall use the following more compact notation for the module actions

$$t\cdot x = \delta_t\cdot x,\quad x\cdot t = x\cdot\delta_t\quad (x\in E,\, t\in S)\,.$$

The standard actions of $\ell^1(S)$ on $\ell^1(S)$ are given by

$$(t\cdot a)(s) = \sum_{u\in[t^{-1}s]} a(u),\quad (a\cdot t)(s) = \sum_{u\in[st^{-1}]} a(u)\quad (s,t\in S,\, a\in\ell^1(S))\,,$$

where again we define the sum over an empty set to be zero. The dual actions of $\ell^1(S)$ on $\ell^\infty(S)$ are given by

$$(t\cdot\lambda)(s) = \lambda(st),\quad (\lambda\cdot t)(s) = \lambda(ts)\quad (s,t\in S,\, \lambda\in\ell^\infty(S))\,.$$

For $s \in S$, the indicator function of the set $\{s\}$ will be denoted by δ_s when considered as an element of $\ell^1(S)$ and by λ_s when considered as an element of $\ell^\infty(S)$. This notation implies that the module actions satisfy $t\cdot\delta_s = \delta_{ts}$ and $t\cdot\lambda_s = \chi_{[st^{-1}]}$ $(t\in S)$.

Proposition 1.7.3 ([5, Theorem 4.6])**.** *Let S be a semigroup. Then $c_0(S)$ is a left [right] $\ell^1(S)$-submodule of $\ell^\infty(S)$ if and only if S is weakly right [left] cancellative.*

Proof. We shall prove that $c_0(S)$ is a left $\ell^1(S)$-submodule of $\ell^\infty(S)$ if and only if S is weakly right cancellative.

First suppose that S is weakly right cancellative. Take $\lambda \in c_0(S)$, $t \in S$ and $\varepsilon > 0$. There is a finite set F with

$$|\lambda(s)| < \varepsilon\quad (s\in S\setminus F)\,.$$

The set Ft^{-1} is finite and

$$|(t \cdot \lambda)(s)| = |\lambda(st)| < \varepsilon \quad (s \in S \setminus Ft^{-1}).$$

Therefore $t \cdot \lambda \in c_0(S)$.

Conversely, suppose that $c_0(S)$ is a left $\ell^1(S)$-submodule of $\ell^\infty(S)$. Take $s, t \in S$. Then $t \cdot \lambda_s = \chi_{[st^{-1}]} \in c_0(S)$. Hence the set $[st^{-1}]$ is finite. This is true for all s and t in S. Therefore S is weakly right cancellative. □

Let S be a semigroup, and let E be a Banach space. We denote by $\delta_s \otimes x$ the function in $\ell^1(S, E)$ which takes the value x at s and is 0 on $S \setminus \{s\}$. We make the identification $\ell^1(S) \widehat{\otimes} E = \ell^1(S, E)$. Thus every element $z \in \ell^1(S) \widehat{\otimes} E$ can be represented as

$$z = \sum_{s \in S} \delta_s \otimes x_s$$

where $(x_s) \subset E$ and $\|z\|_\pi = \sum_{s \in S} \|x_s\|$. The $\ell^1(S)$-module actions on $\ell^1(S, E)$ are determined by

$$t \cdot (\delta_s \otimes x) = \delta_{ts} \otimes x, \quad (\delta_s \otimes x) \cdot t = \delta_{st} \otimes x \quad (s, t \in S,\ x \in E).$$

Suppose that $E \in \ell^1(S)$-**mod**. Then the multiplication map $\pi : \ell^1(S, E) \to E$ satisfies

$$\pi(\delta_s \otimes x) = s \cdot x \quad (s \in S,\ x \in E).$$

Amenability for semigroups and semigroup algebras

We define a *left amenable semigroup* in the same way as Definition 1.5.2 by replacing $L^1(G)$ by $\ell^1(S)$. For a general semigroup S, the condition that S be amenable has fairly weak implications for the Banach algebra $\ell^1(S)$. For example; the condition is necessary for the Banach algebra $\ell^1(S)$ to be amenable but far from sufficient. A characterization of the semigroups such that $\ell^1(S)$ is amenable is given in [5, Theorem 10.12].

Chapter 2

Properties of projective and injective modules

In this chapter we shall study general intrinsic properties of projective and injective modules over a Banach algebra. We also prove some hereditary properties. These results will be applied in later chapters to specific modules and classes of Banach algebras. Most methods of checking projectivity and injectivity are based on the retraction and coretraction problems of Proposition 1.3.2 and Proposition 1.3.5.

We adopt the convention of studying projective left modules and injective right modules. All results have obvious counterparts for modules in other categories.

2.1 Projective modules

Let A be a Banach algebra, and let $E \in A$-**mod** be projective. Let ρ be a left A-module morphism $\rho : E \to A^\flat \widehat{\otimes} E$ with $\pi \circ \rho = I_E$. For each $x \in E$ we can write $\rho(x) = e^\flat \otimes y + z$, where $z \in A \widehat{\otimes} E$. The identity $I_E = \pi \circ \rho$ implies that $y = x - \pi(z)$, and so the *canonical form* of ρ is

$$\rho(x) = e^\flat \otimes (x - \pi(z)) + z\,. \tag{2.1}$$

We shall use this formula in future without further reference.

Proposition 2.1.1. *Let A be a Banach algebra, let $E \in A$-**mod**, and let I be a closed, complemented right ideal in A. Suppose that E is projective. Then there is a projection of E onto $\overline{IE}$.*

Proof. Since E is projective, there exists $\rho \in {}_A\mathcal{B}(E, A^\flat \widehat{\otimes} E)$ with $\pi \circ \rho = I_E$. By Proposition 1.1.7 there is a projection $P : A^\flat \widehat{\otimes} E \to I \widehat{\otimes} E$. Set $Q = \pi \circ P \circ \rho$. We check that this is a projection. Take $x \in \overline{IE}$. We may suppose that $x = a \cdot y$ where $a \in I$ and $y \in E$. Then

$$Q(x) = \pi \circ P(a \cdot \rho(y)) = \pi(a \cdot \rho(y)) = x\,,$$

as required. □

In particular, if $E \in A$-**mod** is projective, then $\overline{AE}$ has a Banach space complement.

Proposition 2.1.2. *Let A be a Banach algebra, and let $E \in A$-**mod** be projective. Suppose that elements $a_1, a_2 \in A$ and $x_1, x_2 \in E$ satisfy $a_1 \cdot x_1 = a_2 \cdot x_2$. Then either the set $\{a_1 + \overline{A^2}, a_2 + \overline{A^2}\}$ is linearly dependent in $A/\overline{A^2}$ or the set $\{x_1 + \overline{AE}, x_2 + \overline{AE}\}$ is linearly dependent in $E/\overline{AE}$.*

Proof. Since E is projective, there exists $\rho \in {}_A\mathcal{B}(E, A^\flat \widehat{\otimes} E)$ with $\pi \circ \rho = I_E$. For $i = 1, 2$, we can write

$$\rho(x_i) = e^\flat \otimes (x_i - \pi(z_i)) + z_i \,,$$

where $z_i \in A \widehat{\otimes} E$. Then

$$0 = \rho(a_1 \cdot x_1 - a_2 \cdot x_2) = a_1 \otimes (x_1 - \pi(z_1)) + a_1 \cdot z_1 - a_2 \otimes (x_2 - \pi(z_2)) - a_2 \cdot z_2 \,.$$

Let $q_A : A \to A/\overline{A^2}$ and $q_E : E \to E/\overline{AE}$ be the quotient maps. Then

$$0 = (q_A \otimes q_E) \circ \rho(a_1 \cdot x_1 - a_2 \cdot x_2) = q_A(a_1) \otimes q_E(x_1) - q_A(a_2) \otimes q_E(x_2) \,,$$

which implies the result. □

2.1.1 The annihilator submodule

Let A be a Banach algebra, and let $E \in A$-**mod**. In this section we shall prove some results about the annihilator submodule ${}^\perp E$.

Proposition 2.1.3. *Let A be a Banach algebra, and let $E \in A$-**mod** be projective.*

(i) *If A does not have a right identity, then ${}^\perp E = {}^\perp(\overline{AE})$.*

(ii) *If A has a unique right identity e, then ${}^\perp E = {}^\perp(\overline{AE}) + \overline{(e^\flat - e) \cdot E}$.*

Proof. We consider the case where A has a unique right identity; the other case is similar. The inclusion ${}^\perp(\overline{AE}) + \overline{(e^\flat - e) \cdot E} \subset {}^\perp E$ is clear.

First we identify ${}^\perp(A^\flat \widehat{\otimes} E)$. Clearly ${}^\perp(A \widehat{\otimes} E) + (e^\flat - e) \widehat{\otimes} E \subset {}^\perp(A^\flat \widehat{\otimes} E)$; we shall show the reverse inclusion.

Take $z \in {}^\perp(A^\flat \widehat{\otimes} E)$. Then we can write $z = e^\flat \otimes y + w$, where $y \in E$ and $w \in A \widehat{\otimes} E$. Let $P : E \to \mathbb{C}y$ be a projection. Then we can write

$$w = a \otimes y + u \,,$$

for some $a \in A$, and $u = (I_A \otimes (I_E - P))(w)$.

If $y = 0$ then $z \in {}^{\perp}(A \widehat{\otimes} E)$. If $y \neq 0$ then there is $\mu \in E'$ with $\langle y, \mu \rangle = 1$ and $\langle x, \mu \rangle = 0$ for $x \in \operatorname{im}(I_E - P)$. Set $T = I_A \otimes \mu : A \widehat{\otimes} E \to A$. For each $b \in A$ we have

$$0 = T(b \cdot z) = T(b \otimes y + ba \otimes y + b \cdot u) = b + b \cdot a\,,$$

since $T(b \cdot u) = 0$. Hence $a = -e$ and we see that $z = (e^\flat - e) \otimes y + u$ with $u \in {}^{\perp}(A \widehat{\otimes} E)$. Hence ${}^{\perp}(A^\flat \widehat{\otimes} E) = {}^{\perp}(A \widehat{\otimes} E) + (e^\flat - e) \widehat{\otimes} E$.

Now, since E is projective, there exists a map $\rho \in {}_A\mathcal{B}(E, A^\flat \widehat{\otimes} E)$ with $\pi \circ \rho = I_E$. Finally, we have

$$\begin{aligned} {}^{\perp}E &= \pi \circ \rho({}^{\perp}E) \subset \pi({}^{\perp}(A^\flat \widehat{\otimes} E)) \\ &= \pi({}^{\perp}(A \widehat{\otimes} E) + (e^\flat - e) \widehat{\otimes} E)) \subset {}^{\perp}(\overline{AE}) + \overline{(e^\flat - e) \cdot E}\,, \end{aligned}$$

as required. □

Lemma 2.1.4. *Let A be a Banach algebra, let $F \in A$-**mod**, and let Y be a Banach space. Suppose that either F or Y has the approximation property and that ${}^{\perp}F$ is complemented. Then ${}^{\perp}(F \widehat{\otimes} Y) = {}^{\perp}F \widehat{\otimes} Y$.*

Proof. Clearly ${}^{\perp}F \widehat{\otimes} Y \subset {}^{\perp}(F \widehat{\otimes} Y)$; we shall show the reverse inclusion.

Let U be a Banach space complement of ${}^{\perp}F$. Then $F \widehat{\otimes} Y = ({}^{\perp}F \widehat{\otimes} Y) \oplus (U \widehat{\otimes} Y)$. Take $z \in {}^{\perp}(F \widehat{\otimes} Y)$, and write $z = z_1 + z_2$ with $z_1 \in {}^{\perp}F \widehat{\otimes} Y$ and $z_2 \in U \widehat{\otimes} Y$. We shall show that $z_2 = 0$. Take $\mu \in Y'$. Since $z_1 \in {}^{\perp}(F \widehat{\otimes} Y)$, we have

$$0 = (I_F \otimes \mu)(a \cdot z_2) = a \cdot (I_F \otimes \mu)(z_2) \quad (a \in A)\,.$$

Since $(I_F \otimes \mu)(z_2) \in U$, we have $(I_F \otimes \mu)(z_2) = 0$. Since this is true for all $\mu \in Y'$ and since F or Y has the approximation property, by Proposition 1.1.8, $z_2 = 0$. Hence ${}^{\perp}(F \widehat{\otimes} Y) = {}^{\perp}F \widehat{\otimes} Y$. □

The following Lemma follows from Proposition 2.1.3(i) since $A^\flat$ is projective in A-**mod**.

Lemma 2.1.5. *Let A be an algebra. Suppose that A does not have a right identity. Then ${}^{\perp}(A^\flat) = {}^{\perp}A$.*

Proof. Clearly ${}^{\perp}A \subset {}^{\perp}(A^\flat)$. Conversely take $a = \alpha e^\flat + b \in (A^\flat)^{\perp}$. Assume that $\alpha \neq 0$. Then $-b/\alpha$ is a right identity for A, a contradiction. Therefore $\alpha = 0$ and $a \in {}^{\perp}A$. □

Proposition 2.1.6. *Let A be a Banach algebra without a right identity, and let $E \in A$-**mod** be projective. Suppose that A or E has the approximation property, and that ${}^{\perp}A$ is a complemented subspace of A. Then ${}^{\perp}E = \overline{{}^{\perp}AE}$.*

Proof. The inclusion $\overline{{}^{\perp}AE} \subset {}^{\perp}E$ is clear.

By Lemmas 2.1.4 and 2.1.5, ${}^{\perp}(A^{\flat} \widehat{\otimes} E) = {}^{\perp}A \widehat{\otimes} E$. Since E is projective, there exists a map $\rho \in {}_A\mathcal{B}(E, A^{\flat} \widehat{\otimes} E)$ with $\pi \circ \rho = I_E$. Then we have

$$ {}^{\perp}E = \pi \circ \rho({}^{\perp}E) \subset \pi({}^{\perp}(A^{\flat} \widehat{\otimes} E)) = \pi({}^{\perp}A \widehat{\otimes} E) \subset \overline{{}^{\perp}AE}\,, $$

as required. □

The case where A does have a right identity is similar. Suppose that A has a unique right identity e. Then ${}^{\perp}A$ is automatically complemented (the map $a \mapsto a - e \cdot a$, $A \to A$ is a projection onto ${}^{\perp}A$). Under the additional hypothesis of Proposition 2.1.6, $E^{\perp} = \overline{{}^{\perp}AE} + \overline{(e^{\flat} - e) \cdot E}$.

2.1.2 Annihilator modules and identities

The sufficiency part of the following proposition is [18, III 1.34].

Proposition 2.1.7. *Let A be a Banach algebra, and let $E \in A$-**mod** be a non-zero annihilator module. Then E is projective if and only if A has a right identity.*

Proof. First suppose that A has a right identity e. Define $\rho : E \to A^{\flat} \widehat{\otimes} E$ by

$$ \rho(x) = (e^{\flat} - e) \otimes x \quad (x \in E)\,. $$

Then ρ is a left A-module morphism with $\pi \circ \rho = I_E$ and so E is projective in A-**mod**.

Conversely, suppose that E is projective. Then there is a left A-module morphism $\rho : E \to A^{\flat} \widehat{\otimes} E$ with $\pi \circ \rho = I_E$. Take $x \in E \setminus \{0\}$ and $\lambda \in E'$ with $\langle x, \lambda \rangle = 1$. Since $\pi(z) = 0$ $(z \in A \widehat{\otimes} E)$, we can write $\rho(x) = e^{\flat} \otimes x + z$ where $z \in A \widehat{\otimes} E$. Set $T = I_{A^{\flat}} \otimes \lambda$. Then $T : A^{\flat} \widehat{\otimes} E \to A^{\flat}$ is a left A-module morphism. Applying T to $a \cdot \rho(x)$ we see that $0 = a + a \cdot T(z)$ $(a \in A)$, and so $-T(z)$ is a right identity for A. □

Proposition 2.1.8. *Let A be a Banach algebra which is faithful in A-**mod**. Suppose that $E \in A$-**mod** is non-faithful and projective, and that either A or E has the approximation property. Then A has a right identity.*

Proof. Since E is projective there is a left A-module morphism $\rho : E \to A^{\flat} \widehat{\otimes} E$ with $\pi \circ \rho = I_E$. Take $x \in E \setminus \{0\}$ with $A \cdot x = 0$. Then $\rho(x) \neq 0$, hence by Proposition 1.1.8 there exists $\lambda \in E'$ with $(I_{A^{\flat}} \otimes \lambda)(\rho(x)) \neq 0$. We can write $(I_{A^{\flat}} \otimes \lambda)(\rho(x)) = \alpha e^{\flat} + b$, where $\alpha \in \mathbb{C}$ and $b \in A$. We have $\alpha a + ab = 0$ $(a \in A)$. If $\alpha = 0$, then since A is left faithful, $b = 0$ which is a contradiction. Hence $\alpha \neq 0$ and $-b/\alpha$ is a right identity for A. □

Proposition 2.1.9. *Let A be a unital Banach algebra, and let $E \in A$-**mod**. Then E is projective if and only if the closed submodule $\overline{AE} = e_A \cdot E$ is projective.*

Proof. The map $x \mapsto e_A \cdot x$, $E \to e_A \cdot E$ is a projection onto $e_A \cdot E$ and a left A-module morphism. Hence

$$E = (e_A \cdot E) \oplus_A E/(e_A \cdot E)\,.$$

By Proposition 2.1.7 the quotient module $E/(e_A \cdot E)$ is projective in A-**mod**. It now follows from [18, VII, Proposition 1.17] that E is projective if and only if $e_A \cdot E$ is projective. □

We now prove analogous results about flat modules.

Proposition 2.1.10. *Let A be a Banach algebra, and let $E \in A$-**mod** be a non-zero annihilator module. Then E is flat if and only if A has a bounded right approximate identity.*

Proof. Let (e_α) be a bounded right approximate identity for A. Define $\rho_\alpha : E \to (A^\flat \widehat{\otimes} E)''$ by

$$\rho_\alpha(x) = (e^\flat - e_\alpha) \otimes x \quad (x \in E)\,.$$

Regard (ρ_α) as a bounded net in $\mathcal{B}(E, (A^\flat \widehat{\otimes} E)'') = \big(E \widehat{\otimes} (A^\flat \widehat{\otimes} E)'\big)'$. Let ρ be a weak-$*$ accumulation point of this net. Then ρ is a left A-module morphism such that $\pi'' \circ \rho = \iota_E$. Therefore E is flat in A-**mod**.

Conversely, suppose that E is flat. Then there is a left A-module morphism $\rho : E \to (A^\flat \widehat{\otimes} E)''$ with $\pi'' \circ \rho = \iota_E$. Take $x \in E \setminus \{0\}$ and $\lambda \in E'$ with $\langle x, \lambda \rangle = 1$. We make the identification

$$(A^\flat \widehat{\otimes} E)'' = \mathbb{C}e^\flat \widehat{\otimes} E'' \oplus_1 (A \widehat{\otimes} E)''\,.$$

Since $\pi''(z) = 0 \ \ (z \in (A \widehat{\otimes} E)'')$, we can write $\rho(x) = e^\flat \otimes \iota_E(x) + z$ where $z \in (A \widehat{\otimes} E)''$. Set $T = I_{A^\flat} \otimes \lambda \in \mathcal{B}(A^\flat \widehat{\otimes} E, E)$. Then $T'' \in \mathcal{B}((A^\flat \widehat{\otimes} E)'', E'')$ is a left A-module morphism. Applying T'' to $a \cdot \rho(x)$ we see that $0 = \iota_A(a) + a \cdot T''(z) \ \ (a \in A)$. Set $\Phi = -T(z) \in A''$. Then $a \cdot \Phi = \iota_A(a) \ \ (a \in A)$. This is equivalent to A having a bounded right approximate identity, (see [4, 2.9.16]). □

Proposition 2.1.11. *Let A be a Banach algebra with a bounded approximate identity, and let $E \in A$-**mod**. Then E is flat if and only if the closed submodule $\overline{AE} = A \cdot E$ is flat.*

Proof. We borrow part of the argument from [4, 2.9.26]. Let (e_α) be a bounded approximate identity for A, and define a net $T_\alpha \subset \mathcal{B}(E') = (E' \widehat{\otimes} E)'$ by

$$T_\alpha(\lambda) = \lambda \cdot e_\alpha \quad (\lambda \in E')\,.$$

Let T be a weak-$*$ accumulation point of this bounded net. Then $T - I_{E'}$ is a projection from E' onto $(A \cdot E)^0$ and a right A-module morphism. Hence

$$E' = (A \cdot E)^0 \oplus_A E'/(A \cdot E)^0 = (E/A \cdot E)' \oplus_A (A \cdot E)' .$$

By the above proposition the quotient module $E/A \cdot E$ is flat in A-**mod**. It now follows from [18, VII, Proposition 1.17] that E is flat if and only if $A \cdot E$ is flat. □

2.2 Injective modules

Let A be a Banach algebra, and let $E \in$ **mod**-A be injective. Then there exists a right A-module morphism $\rho : \mathcal{B}(A^\flat, E) \to E$ with $\rho \circ \Pi = I_E$. Let $P_A, P_{\mathbb{C}e^\flat} \in \mathcal{B}(A^\flat)$ be projections on to the subspaces A and $\mathbb{C}e^\flat$, respectively. Each $T \in \mathcal{B}(A^\flat, E)$ can be written as $T = T \circ P_A + T \circ P_{\mathbb{C}e^\flat}$. We have $T \circ P_{\mathbb{C}e^\flat} = (\Pi x_T) \circ P_{\mathbb{C}e^\flat}$, where $x_T = T(e^\flat)$. The identity $I_E = \rho \circ \Pi$ gives $\rho((\Pi x) \circ P_{\mathbb{C}e^\flat}) = x - \rho((\Pi x) \circ P_A)$ $(x \in E)$, and hence the *canonical form* of ρ is

$$\rho(T) = \rho(T \circ P_A) + x_T - \rho((\Pi x_T) \circ P_A) , \tag{2.2}$$

where $x_T = T(e^\flat)$. We shall use this formula later without further reference.

Proposition 2.2.1. *Let A be a Banach algebra, let $E \in$ **mod**-A, and let I be a closed, complemented right ideal in A. Suppose that E is injective. Then there is a projection of E onto $E^{\perp I}$.*

Proof. Since E is injective, there exists $\rho \in \mathcal{B}_A(\mathcal{B}(A^\flat, E), E)$ with $\rho \circ \Pi = I_E$. For $x \in E$ set

$$Q(x) = x - \rho\left(\Pi(x) \circ P\right) ,$$

where $P \in \mathcal{B}(A^\flat)$ is a projection on to I. We claim that Q is a projection onto $E^{\perp I}$. Indeed, since $(\Pi(x) \circ P) \cdot a = \Pi(x \cdot a)$ $(a \in I)$, we have

$$Q(x) \cdot a = x \cdot a - \rho(\Pi(x) \circ P) \cdot a = x \cdot a - \rho(\Pi(x \cdot a)) = 0 \quad (a \in I) ,$$

so that $Q(E) \subset E^{\perp I}$. Finally, for $x \in E^{\perp I}$ we have $\Pi(x) \circ P = 0$, and hence Q is a projection. □

Corollary 2.2.2. *Let A be a Banach algebra, let $E \in A$-**mod**, and let I be a closed, complemented right ideal in A. Suppose that E is flat. Then $\overline{IE}$ is weakly complemented in E.*

Proof. This follows from Proposition 2.2.1 since

$$(E')^{\perp I} = \{\lambda \in E' : \lambda \cdot I = 0\} = \overline{IE}^{0} .$$ □

Proposition 2.2.3. *Let A be a Banach algebra, and let $E \in$ **mod**-A be injective. Suppose that there exists $a_0 \in A \setminus \{0\}$ with $a_0 A = 0$. Then $E^\perp = E \cdot a_0$. In particular, the subspace $E \cdot a_0$ is closed.*

Proof. We first prove the result for the right A-module $\mathcal{B}(A^\flat, E)$. Take $\mu \in (A^\flat)'$ with $\langle a_0, \mu \rangle = 1$. Take $T \in \mathcal{B}(A^\flat, E)$ with $T|A = 0$. Then

$$T = (T(e^\flat) \otimes \mu) \cdot a_0$$

where $T(e^\flat) \otimes \mu \in \mathcal{B}(A^\flat, E)$. Hence we have

$$\mathcal{B}(A^\flat, E)^\perp = \big\{T \in \mathcal{B}(A^\flat, E) : T|A = 0\big\} = \mathcal{B}(A^\flat, E) \cdot a_0 \,.$$

Let $\rho \in \mathcal{B}_A(E, \mathcal{B}(A^\flat, E))$ with $\rho \circ \Pi = I_E$. Then we have

$$E^\perp = \rho \circ \Pi(E^\perp) \subset \rho(\mathcal{B}(A^\flat, E)^\perp) = \rho(\mathcal{B}(A^\flat, E) \cdot a_0) \subset E \cdot a_0 \,.$$

The reverse inclusion $E \cdot a_0 \subset E^\perp$ is clear. □

Example 2.2.4 ([34, Example 4.4])**.** Let X be a Banach space with $\dim X \geq 2$, and take $\varphi \in X' \setminus \{0\}$. Define a product on X by

$$ab = \varphi(a)b \quad (a, b \in X)\,.$$

With this product X is a Banach algebra which, following [34], we denote by $A_\varphi(X)$. Take any $a_1 \in A_\varphi(X)$ with $\langle a_1, \varphi \rangle = 1$ and define $\rho : A_\varphi(X) \to A_\varphi(X) \widehat{\otimes} A_\varphi(X)$ by $\rho(a) = a_1 \otimes a$ $(a \in A_\varphi(X))$. It is easily checked that ρ is a bimodule morphism, and so $A_\varphi(X)$ is biprojective. Note that $A_\varphi(X)$ does not have a right identity, but any a with $\langle a, \varphi \rangle = 1$ is a left identity. Since $A_\varphi(X)$ does not have a right identity it is not left injective. The Banach algebra $A_\varphi(X)$ is faithful in $A_\varphi(X)$-**mod**, but not faithful in **mod**-$A_\varphi(X)$.

Proposition 2.2.5. *The module $A_\varphi(X)$ is injective in* **mod**-*$A_\varphi(X)$ if and only if*

$$\dim X = 2\,.$$

Proof. Set $A = A_\varphi(X)$.

Choose $a_0 \in \ker \varphi \setminus \{0\}$. By Proposition 2.2.3 we have

$$\ker \varphi = A^\perp = A a_0 = \mathbb{C} a_0 \,.$$

By the rank-nullity theorem, $\dim X = 2$.

Conversely, suppose that $\dim X = 2$. Let $\{a_0, a_1\}$ be a basis of X with $\langle a_0, \varphi \rangle = 0$ and $\langle a_1, \varphi \rangle = 1$. Take $\lambda \in X'$ with $\langle a_0, \lambda \rangle = 1$ and $\langle a_1, \lambda \rangle = 0$. We define a map $\rho : \mathcal{B}(A^\flat, A) \to A$ by

$$\rho(T) = \langle T a_0, \lambda \rangle\, a_1 - \langle T a_1, \lambda \rangle\, a_0 + \big\langle T e^\flat, \lambda \big\rangle\, a_0 \quad (T \in \mathcal{B}(A^\flat, A))\,.$$

This is a right A-module morphism with $\rho \circ \Pi = I_A$. The easiest way to see this is to check that $\rho(T \cdot a_0) = \rho(T) \cdot a_0$, $\rho(T \cdot a_1) = \rho(T) \cdot a_1$, $\rho(\Pi a_0) = a_0$, and $\rho(\Pi a_1) = a_1$. Therefore A is injective in **mod**-A. $\square$

The following generalizes [8, Proposition 1.8].

Proposition 2.2.6. *Let A be a Banach algebra, and let $E \in$ **mod**-A be injective. Suppose that there exists $Q \in \mathcal{B}(A, E)$ and a subset $S \subset A$ such that*

$$Q(ba) = Q(b) \cdot a \quad (b \in S,\, a \in A)\,.$$

Then there is an element $x \in E$ with

$$Q(b) = x \cdot b \quad (b \in S)\,.$$

Proof. There exists a morphism $\rho \in \mathcal{B}_A(\mathcal{B}(A^\flat, E), E)$ such that $\rho \circ \Pi = I_E$. Extend Q to $A^\flat$ by setting $Q(e^\flat) = 0$. Then $Q \in \mathcal{B}(A^\flat, E)$ and

$$Q \cdot b = \Pi(Q(b)) \quad (b \in S)\,.$$

Set $x = \rho(Q) \in E$. Then we have

$$Q(b) = (\rho \circ \Pi)(Q(b)) = \rho(Q \cdot b) = x \cdot b \quad (b \in S)\,,$$

as required. $\square$

Corollary 2.2.7. *Let A be a Banach algebra, let I be a complemented ideal in A, and let $E \in$ **mod**-A be injective. Then the map $j : E \to \mathcal{B}_A(I, E)$ given by*

$$j(x) : a \mapsto x \cdot a,\; I \to E$$

is a Banach A-module epimorphism with kernel $E^{\perp I}$.

Proof. Take $T \in \mathcal{B}_A(I, E)$, and set $Q = T \circ P$, where $P : A^\flat \to I$ is a projection. Then $Q \in \mathcal{B}(A^\flat, E)$ and satisfies $Q(ba) = Q(b) \cdot a$ $(b \in I, a \in A)$. By Proposition 2.2.6, there exists $x \in E$ such that $T(b) = Q(b) = x \cdot b$ $(b \in I)$, i.e., $T = j(x)$.

The rest is clear. $\square$

Corollary 2.2.8. *Let A be a Banach algebra.*

(i) *Let A be a subalgebra of a Banach algebra B. Suppose that B is injective in* **mod**-A. *Then there exists $b \in B$ with $ba = a$ $(a \in A)$.*

(ii) *Let I be a closed right ideal in A. Suppose that A/I is injective in* **mod**-A. *Then I has a left modular identity.*

(iii) *Let I be a closed, complemented right ideal in A. Suppose that I is injective in* **mod**-A. *Then there exists $b \in I$ with $ba = a$ $(a \in I)$.*

(iv) *Let I be a closed, complemented right ideal in A. Suppose that I'' is injective in* **mod**-A. *Then I has a bounded left approximate identity.*

Proof. These all follow from Proposition 2.2.6 by choosing specific maps Q.

(i) Take $Q : A \to B$ to be the inclusion map.

(ii) Take $Q : A \to A/I$ to be the quotient map.

(iii) Take $Q : A \to I$ to be a projection.

(iv) Take the map $Q : A \to I''$ to be a projection onto I followed by the natural embedding into I''. Then there exists $\Phi \in I''$ with $\Phi \cdot a = \iota_I(a)$ $(a \in I)$. This is equivalent to I having a bounded left approximate identity. □

2.2.1 Amenability and injectivity

This section is a generalization of the argument in [8, §4].

Definition 2.2.9. Let A be a Banach algebra, and let $E \in \textbf{mod}\text{-}A$. Then an element $\lambda \in E' \setminus \{0\}$ is an *augmentation-invariant* functional if there exists a character φ on A, with $a \cdot \lambda = \varphi(a)\lambda$ for each $a \in A$. The triple (E, λ, φ) is an *augmentation-invariant* Banach right A-module.

Example 2.2.10. (i) Let A be a Banach algebra with a character φ, and let I be a closed right ideal of A with $I \not\subset \ker \varphi$. Then $(I, \varphi|I, \varphi)$ is an augmentation-invariant right A-module.

(ii) If $E \in \textbf{mod}\text{-}A$ is augmentation-invariant, then so is E''.

Lemma 2.2.11. *Let A be a Banach algebra, let I be a complemented right ideal of A, and let (E, λ, φ) be an augmentation-invariant Banach right A-module with $I \not\subset \ker \varphi$. Suppose that E is injective in* **mod**-A. *Then there exists $\Lambda \in \mathcal{B}(I, E)'$ such that:*

(i) $a \cdot \Lambda = \varphi(a)\Lambda$ *for each* $a \in A$;

(ii) $\langle \Pi(x), \Lambda \rangle = \langle x, \lambda \rangle$ *for each* $x \in E$.

Proof. Since E is injective there is a right A-module morphism $\rho : \mathcal{B}(A^\flat, E) \to E$ with $\rho \circ \Pi = I_E$. Set $\Lambda_0 = \rho'(\lambda) \in \mathcal{B}(A^\flat, E)'$. Since ρ' is a left A-module morphism, we have $a \cdot \Lambda_0 = \varphi(a)\Lambda_0$ $(a \in A)$.

Take $S \in \mathcal{B}(A^\flat, E)$ such that $S|I = 0$. Pick $a_0 \in I$ with $\varphi(a_0) = 1$. Then $S \cdot a_0 = 0$, and

$$0 = \langle S \cdot a_0, \Lambda_0 \rangle = \langle S, \Lambda_0 \rangle \ .$$

Now take $T \in \mathcal{B}(I, E)$. We can extend T to $\widetilde{T} \in \mathcal{B}(A^\flat, E)$ by setting $\widetilde{T} = T \circ P$, where $P : A^\flat \to I$ is a projection. Set

$$\langle T, \Lambda \rangle := \left\langle \widetilde{T}, \Lambda_0 \right\rangle \quad (T \in \mathcal{B}(I, E)) \, .$$

Since $\left(\widetilde{T\cdot a} - \widetilde{T}\cdot a\right)|I = 0 \ (a \in A)$, it follows that $a \cdot \Lambda = \varphi(a)\Lambda \ (a \in A)$. Similarly, since $\left(\widetilde{\Pi(x)} - \Pi(x)\right)|I = 0 \ (x \in E)$, it follows that $\langle \Pi(x), \Lambda\rangle = \langle x, \lambda\rangle$. □

In the following theorem we set $\widetilde{\pi} = \Pi' : \mathcal{B}(I,E)' = (I \widehat{\otimes} F)'' \to E' = F''$. If F is a left A-submodule of F'' then $\widetilde{\pi}|I \widehat{\otimes} F \subset F$, and we can replace $\widetilde{\pi}$ by π.

Theorem 2.2.12. *Let A be a Banach algebra, let I be a complemented right ideal of A, and let (E, λ, φ) be an augmentation-invariant Banach right A-module with $I \not\subset \ker\varphi$. Suppose that E is the dual of a Banach space F, and that E is injective in* **mod**-*A. Then there exists a bounded net $(v_\alpha) \subset I \widehat{\otimes} F$ such that:*

(i) $\lim_\alpha \|a \cdot v_\alpha - \varphi(a)v_\alpha\|_\pi = 0$ *for each $a \in A$;*

(ii) $\lim_\alpha \langle x, \widetilde{\pi}(v_\alpha)\rangle = \langle x, \lambda\rangle$ *for each $x \in E$.*

Proof. Set $X = I \widehat{\otimes} F$, and let $\sigma = \sigma(X, X')$ be the weak topology on X.

First, a net (u_α) is indexed by the family of all finite subsets of $\mathcal{B}(I,E)$, with the ordering specified by inclusion. For each such $\alpha = \{T_1, \ldots, T_k\}$, choose $u_\alpha \in X$ such that $\langle T_i, u_\alpha\rangle = \langle T_i, \Lambda\rangle \ (i = 1, \ldots, k)$, where $\Lambda \in X''$ was specified in Lemma 2.2.11.

For each $a \in A$ and $T \in \mathcal{B}(I,E)$, we have

$$\langle T, a \cdot u_\alpha\rangle = \langle T \cdot a, u_\alpha\rangle = \langle T \cdot a, \Lambda\rangle = \varphi(a)\langle T, \Lambda\rangle = \varphi(a)\langle T, u_\alpha\rangle\,,$$

for each sufficiently large α, and so $\lim_\alpha(a \cdot u_\alpha - \varphi(a)u_\alpha) = 0$ in (X, σ).

Also for each $x \in E$, we have

$$\langle x, \widetilde{\pi}(u_\alpha)\rangle = \langle \Pi(x), u_\alpha\rangle = \langle \Pi(x), \Lambda\rangle = \langle x, \lambda\rangle\,,$$

for each sufficiently large α, and so $\lim_\alpha \langle \Pi(x), u_\alpha\rangle - \langle x, \lambda\rangle = 0$.

Let $\{a_1, \ldots, a_k\}$ and $\{x_1, \ldots, x_\ell\}$ be finite subsets of A and E, respectively, and let $\varepsilon > 0$. Let

$$C = C(\{x_1, \ldots, x_\ell\}, \varepsilon) = \{z \in X : |\langle \Pi(x_i), z\rangle - \langle x_i, \lambda\rangle| < \varepsilon \ (i = 1, \ldots, \ell)\}\,,$$

and consider the Banach space $Y = X_1 \oplus \cdots \oplus X_k$, where each $X_i = X \ (i = 1, \ldots, k)$ and we are taking the ℓ^1-sum. Also consider the linear operator

$$W : z \mapsto (a_1 \cdot z - \varphi(a_1)z, \ldots, a_k \cdot z - \varphi(a_k)z), \quad X \to Y\,.$$

The set C is convex in X, and so $W(C)$ is convex in Y. We have shown that 0 belongs to the $\sigma(Y, Y')$-closure of $W(C)$ in Y. By Mazur's theorem, it follows that 0 belongs to the $\|\cdot\|$-closure of $W(C)$ in Y. The existence of the required net (v_α) follows. □

2.3 Hereditary properties

In this section we shall prove a variety of hereditary type properties for projectivity and injectivity. We do not aim for the most general results, only what we shall need in later chapters.

Banach $*$-modules

Banach $*$-modules have an obvious symmetry between left and right. Let A be a Banach algebra, and let $E \in A$-**mod**-A. We denote by E_L and E_R the left and right A-modules obtained by restricting the action of A to one side.

Proposition 2.3.1. *Let A be a Banach $*$-algebra, and let $E \in A$-**mod**-A be a Banach $*$-module. Then E_L is projective [injective] in A-**mod** if and only if E_R is projective [injective] in **mod**-A.*

Proof. We give the proof only for projectivity.

Suppose that E_L is projective in A-**mod**. We shall prove that E_R is projective in **mod**-A. We define the following maps

$$\pi_L : A^\flat \widehat{\otimes} E \to E_L \quad \text{by} \quad a \otimes x \mapsto a \cdot x \quad (a \in A^\flat,\, x \in E)$$

and

$$\pi_R : A^\flat \widehat{\otimes} E \to E_R \quad \text{by} \quad a \otimes x \mapsto x \cdot a \quad (a \in A^\flat,\, x \in E)\,.$$

By Proposition 1.2.5 we can equip the space $A^\flat \widehat{\otimes} E$ with an involution $*$ given by

$$(a \otimes x)^* = a^* \otimes x^* \quad (a \in A^\flat,\, x \in E)\,.$$

This involution satisfies $(a \cdot z)^* = z^* \cdot a^*$ $(a \in A, z \in A^\flat \widehat{\otimes} E)$. Since E_L is projective in A-**mod** there exists $\rho_L \in {}_A\mathcal{B}(E_L, (A^\flat \widehat{\otimes} E)_L)$ with $\pi_L \circ \rho_L = I_E$. Define $\rho_R : E \to A^\flat \widehat{\otimes} E$ by

$$\rho_R(x) = \rho_L(x^*)^* \quad (x \in E)\,.$$

Then $\rho_R \in \mathcal{B}(E, A^\flat \widehat{\otimes} E)$. For $a \in A$ and $x \in E$ we have

$$\rho_R(x \cdot a) = \rho_L(a^* \cdot x^*)^* = (a^* \cdot \rho_L(x^*))^* = \rho_R(x) \cdot a\,,$$

hence $\rho_R \in \mathcal{B}_A(E_R, (A^\flat \widehat{\otimes} E)_R)$. Since

$$\pi_R(z) = \pi_L(z^*)^* \quad (z \in A^\flat \widehat{\otimes} E)\,,$$

we have $\pi_R \circ \rho_R = I_E$. Therefore E_R is projective in **mod**-A. □

Closed ideals

Part (i) of the following lemma is given in [18, IV 2.4].

Lemma 2.3.2. *Let A be a Banach algebra, and let $E, F \in A$-**mod**. Suppose that either*

(i) *I is a closed left ideal of A and E is essential in I-**mod**, or*

(ii) *I is a closed right ideal of A and F is faithful in I-**mod**.*

Then ${}_A\mathcal{B}(E,F) = {}_I\mathcal{B}(E,F)$.

Proof. Clearly ${}_A\mathcal{B}(E,F) \subset {}_I\mathcal{B}(E,F)$. Take $\varphi \in {}_I\mathcal{B}(E,F)$, $x \in E$, and $b \in A$.

(i) Suppose that $x = a \cdot y$ for some $a \in I$ and $y \in E$. Then

$$\varphi(b \cdot x) = \varphi(ba \cdot y) = ba \cdot \varphi(y) = b \cdot \varphi(x)\,.$$

This equation now follows for all $x \in E$ since $E = \overline{IE}$. Therefore $\varphi \in {}_A\mathcal{B}(E,F)$.

(ii) For each $a \in I$, we have

$$a \cdot (\varphi(b \cdot x) - b \cdot \varphi(x)) = \varphi(ab \cdot x) - ab \cdot \varphi(x) = 0\,.$$

Since F is faithful in I-**mod**, $\varphi(b \cdot x) - b \cdot \varphi(x) = 0$. Therefore $\varphi \in {}_A\mathcal{B}(E,F)$. $\square$

Proposition 2.3.3. *Let A be a Banach algebra, let I be a closed subalgebra of A, and let $E \in A$-**mod** be essential in I-**mod**.*

(i) *Suppose that I is a left ideal and that E is projective in I-**mod**. Then E is projective in A-**mod**.*

(ii) *Conversely, suppose that I is a complemented right ideal and that E is projective in A-**mod**. Then E is projective in I-**mod**.*

Proof. (i) Since E is projective in I-**mod** there exists $\rho \in {}_I\mathcal{B}(E, I \widehat{\otimes} E)$ with $\pi_{I \widehat{\otimes} E} \circ \rho = I_E$. Let $i : I \to A$ be the natural embedding and set $\widetilde{\rho} = (i \otimes I_E) \circ \rho$. By Lemma 2.3.2(i), $\widetilde{\rho} \in {}_A\mathcal{B}(E, A \widehat{\otimes} E)$. Since $\pi_{A \widehat{\otimes} E} \circ (i \otimes I_E) = \pi_{I \widehat{\otimes} E}$ we have $\pi_{A \widehat{\otimes} E} \circ \widetilde{\rho} = I_E$. Therefore E is projective in A-**mod**.

(ii) Conversely, since E is projective in A-**mod** there exists $\rho \in {}_A\mathcal{B}(E, A \widehat{\otimes} E)$ with $\pi_{A \widehat{\otimes} E} \circ \rho = I_E$. Since I is complemented, by Proposition 1.1.7, $I \widehat{\otimes} E$ is a closed complemented subspace of $A \widehat{\otimes} E$. Since E is essential in I-**mod**, $\rho(E) \subset I \widehat{\otimes} E$. Therefore E is projective in I-**mod**. $\square$

Proposition 2.3.4. *Let A be a Banach algebra, let I be a closed subalgebra of A, and let $E \in$ **mod**-A be faithful in **mod**-I.*

(i) *Suppose that I is a left ideal and E is injective in **mod**-I. Then E is injective in **mod**-A.*

(ii) *Conversely, suppose that I is a complemented right ideal and E is injective in **mod**-A. Then E is injective in **mod**-I.*

Proof. (i) Since E is injective in **mod**-I there exists $\rho \in \mathcal{B}_I(\mathcal{B}(I,E),E)$ with $\rho \circ \Pi_{\mathcal{B}(I,E)} = I_E$. Define $\widetilde{\rho} : \mathcal{B}(A,E) \to E$ by

$$\widetilde{\rho}(U) = \rho(U|I) \quad (U \in \mathcal{B}(A,E)).$$

Then by (a right module version of) Lemma 2.3.2(ii), $\widetilde{\rho} \in \mathcal{B}_A(\mathcal{B}(A,E),E)$. Since $\Pi_{\mathcal{B}(A,E)}|I = \Pi_{\mathcal{B}(I,E)}$, we have $\widetilde{\rho} \circ \Pi = I_E$. Therefore E is injective in A-**mod**.

(ii) Conversely, since E is injective in **mod**-A there exists $\rho \in \mathcal{B}_A(\mathcal{B}(A,E),E)$ with $\rho \circ \Pi = I_E$.

Take $S \in \mathcal{B}(A,E)$ such that $S|I = 0$. Then $S \cdot a = 0 \ (a \in I)$, and so

$$\rho(S) \cdot a = \rho(S \cdot a) = 0 \quad (a \in I).$$

Since E is faithful in **mod**-I, $\rho(S) = 0$. Now take $U \in \mathcal{B}(I,E)$, and extend U to $\widetilde{U} \in \mathcal{B}(A,E)$ by setting $\widetilde{U} = U \circ P$ where $P : A \to I$ is a projection. Define $\widetilde{\rho} : \mathcal{B}(I,E) \to E$ by $\widetilde{\rho}(U) = \rho(\widetilde{U}) \ (U \in \mathcal{B}(I,E))$. Then for $U \in \mathcal{B}(I,E)$ and $a \in A$ we have

$$\widetilde{\rho}(U \cdot a) - \widetilde{\rho}(U) \cdot a = \rho(\widetilde{U \cdot a} - \widetilde{U} \cdot a) = 0$$

because $\left(\widetilde{U \cdot a} - \widetilde{U} \cdot a\right)|I = 0$. Hence $\widetilde{\rho} \in \mathcal{B}_I(\mathcal{B}(I,E),E)$. Further, $\widetilde{\rho} \circ (\Pi|I) = I_E$ since $\left(\Pi - \widetilde{\Pi|I}\right)|I = 0$. Therefore E is injective in **mod**-I. □

Direct Sums

Let A and B be Banach algebras, and let $C > 0$. Then B is a C-*retract* of A if there exist Banach algebra homomorphisms $\theta : B \to A$ and $\tau : A \to B$ with $\tau \circ \theta = I_B$ and $\|\tau\| \, \|\theta\| \le C$. The following can be proved by an easy diagram chase.

Lemma 2.3.5. *Let B be a C_1-retract of a Banach algebra A. Suppose that A is C_2-left projective [flat]. Then B is C_1C_2-left projective [flat].* □

Proposition 2.3.6. *Let $\{A_\lambda : \lambda \in \Lambda\}$ be a collection of Banach algebras.*

(i) *Suppose that each $A_\lambda \ (\lambda \in \Lambda)$ is essential. Then ℓ^1-$\bigoplus\{A_\lambda : \lambda \in \Lambda\}$ is left projective [flat] if and only if there is a constant $C > 0$ such that each $A_\lambda \ (\lambda \in \Lambda)$ is C-left projective [flat].*

(ii) *Suppose that each $A_\lambda \ (\lambda \in \Lambda)$ is right faithful. Then ℓ^1-$\bigoplus\{A_\lambda : \lambda \in \Lambda\}$ is right injective if and only if Λ is finite and each $A_\lambda \ (\lambda \in \Lambda)$ is left injective.*

Proof. Set $A = \ell^1$-$\bigoplus\{A_\lambda : \lambda \in \Lambda\}$.

(i) We give the proof for projectivity. First suppose that A is left projective. Then A is C-left projective for some $C > 0$. For each $\lambda \in \Lambda$ the Banach algebra A_λ is a 1-retract of A. Hence by Lemma 2.3.5 A_λ is C-left projective.

Conversely, suppose there is a constant $C > 0$ such that each A_λ $(\lambda \in \Lambda)$ is C-left projective. Then for each $\lambda \in \Lambda$ there is a left A_λ-module morphism $\rho_\lambda : A_\lambda \to A_\lambda \widehat{\otimes} A_\lambda$ with $\pi_\lambda \circ \rho_\lambda = I_{A_\lambda}$, and $\|\rho_\lambda\| \le C$. Since A_λ is a 1-complemented subspace of A by Proposition 1.1.7 $A_\lambda \widehat{\otimes} A_\lambda$ is a 1-complemented subspace of $A \widehat{\otimes} A$. For $a = \sum_\lambda a_\lambda \delta_\lambda \in A$ we define $\rho(a) \in A \widehat{\otimes} A$ by

$$\rho(a) = \sum_\lambda \rho_\lambda(a_\lambda)\,.$$

We have

$$\|\rho(a)\|_{A \widehat{\otimes} A} \le \sum_\lambda \|\rho_\lambda(a_\lambda)\|_{A \widehat{\otimes} A} = \sum_\lambda \|\rho_\lambda(a_\lambda)\|_{A_\lambda \widehat{\otimes} A_\lambda} \le \sum_\lambda C\,\|a_\lambda\|_{A_\lambda} = C\,\|a\|_A\ ,$$

hence ρ is bounded with $\|\rho\| \le C$. Clearly ρ is a left A-module morphism with $\pi_A \circ \rho = I_A$. Therefore A is left projective.

(ii) First suppose that A is right injective. Then by Corollary 2.2.8(i) A has a left identity $e_A = \sum_\lambda e_\lambda \delta_\lambda$. For each $\lambda \in \Lambda$, e_λ is a left identity for A_λ, and so $\|e_\lambda\|_{A_\lambda} \ge 1$. Since $\|e_A\| = \sum_\lambda \|e_\lambda\|_{A_\lambda}$, the set Λ must be finite. It is easily checked by diagram chasing that each A_λ is right injective.

Conversely, suppose that Λ is finite and each A_λ $(\lambda \in \Lambda)$ is right injective. Then, for each $\lambda \in \Lambda$, there exists a right A_λ-module morphism $\rho_\lambda : \mathcal{B}(A_\lambda, A_\lambda) \to A_\lambda$ with $\rho_\lambda \circ \Pi_\lambda = I_{A_\lambda}$. For $T \in \mathcal{B}(A, A)$ we define $T_\lambda \in \mathcal{B}(A_\lambda, A_\lambda)$ by

$$T_\lambda(a_\lambda) = (P_\lambda \circ T)(a_\lambda \delta_\lambda) \quad (a_\lambda \in A_\lambda)\,,$$

where $P_\lambda : A \to A_\lambda$ is a projection. For $a = \sum_\lambda a_\lambda \delta_\lambda \in A$ we have $(T \cdot a)_\lambda = T_\lambda \cdot a_\lambda$. Now define $\rho : \mathcal{B}(A, A) \to A$ by

$$\rho(T) = \sum_\lambda \rho_\lambda(T_\lambda)\delta_\lambda\,.$$

For $a \in A$ and $T \in \mathcal{B}(A, A)$ we have

$$\rho(T \cdot a) = \sum_\lambda \rho_\lambda((T \cdot a)_\lambda) = \sum_\lambda \rho_\lambda(T_\lambda \cdot a_\lambda) = \sum_\lambda \rho_\lambda(T_\lambda) \cdot a_\lambda = \rho(T) \cdot a\,.$$

Hence ρ is a right A-module morphism and $\rho \circ \Pi = \sum_\lambda \rho_\lambda \circ \Pi_\lambda = \sum_\lambda I_{A_\lambda} = I_A$. Therefore A is right injective. □

The Banach algebra $A = \ell^1\text{-}\bigoplus\{A_\lambda : \lambda \in \Lambda\}$ need not be left projective if some of the Banach algebras A_λ are not essential. Indeed, suppose that the direct sum A contains two non-essential Banach algebras A_λ and A_μ. Assume towards a contradiction that A is left projective. For $a \in A_\lambda \setminus \overline{A_\lambda^2}$ and $b \in A_\mu \setminus \overline{A_\mu^2}$ we have $(a\delta_\lambda)(b\delta_\mu) - (b\delta_\mu)(a\delta_\lambda) = 0$. By Proposition 2.1.2 the set $\{a\delta_\lambda + \overline{A^2}, b\delta_\mu + \overline{A^2}\}$ is linearly dependent in $A/\overline{A^2}$, which is a contradiction.

The Banach algebra $A = \ell^1\text{-}\bigoplus\{A_\lambda : \lambda \in \Lambda\}$ need not be right injective if some of the Banach algebras A_λ are not right faithful. Indeed, suppose that the direct sum A contains two non-right faithful Banach algebras A_λ and A_μ. Assume towards a contradiction that A is right injective. Take $a \in A_\lambda \setminus \{0\}$ with $aA_\lambda = 0$ and $b \in A_\mu \setminus \{0\}$ with $bA_\mu = 0$. We have $a\delta_\lambda A = 0$ and $b\delta_\mu A = 0$. By Proposition 2.2.3 $A^\perp = Aa\delta_\lambda = Ab\delta_\mu$. This can happen only if $A_\lambda a = 0$ and $A_\mu b = 0$. This is impossible since A_λ and A_μ are both right injective, and hence have left identities.

Proposition 2.3.7. *Let $\{A_\lambda : \lambda \in \Lambda\}$ be a collection of Banach algebras. Then $\ell^1\text{-}\bigoplus\{A_\lambda : \lambda \in \Lambda\}$ is [biprojective / biflat] if and only if there is a constant $C > 0$ such that each A_λ $(\lambda \in \Lambda)$ is [C-biprojective / C-biflat].*

Proof. Since a biflat Banach algebra is essential this follows in exactly the same way as Proposition 2.3.6. □

Tensor products

Proposition 2.3.8. *Let A be a C_1-biprojective Banach algebra, and let B be a C_2-biprojective Banach algebra. Then $A \widehat{\otimes} B$ is C_1C_2-biprojective.*

Proof. There exist an A-bimodule morphism $\rho_A : A \to A \widehat{\otimes} A$ with $\pi_A \circ \rho_A = I_A$ and $\|\rho_A\| \le C_1$ and a B-bimodule morphism $\rho_B : B \to B \widehat{\otimes} B$ with $\pi_B \circ \rho_B = I_B$ and $\|\rho_B\| \le C_2$. Let $\theta : (A \widehat{\otimes} A) \widehat{\otimes} (B \widehat{\otimes} B) \to (A \widehat{\otimes} B) \widehat{\otimes} (A \widehat{\otimes} B)$ be the isometric isomorphism given by

$$a_1 \otimes a_2 \otimes b_1 \otimes b_2 \mapsto a_1 \otimes b_1 \otimes a_2 \otimes b_2 \quad (a_1, a_2 \in A,\ b_1, b_2 \in B)\,.$$

We set

$$\rho = \theta \circ (\rho_A \otimes \rho_B) : A \widehat{\otimes} B \to (A \widehat{\otimes} B) \widehat{\otimes} (A \widehat{\otimes} B)\,.$$

It is easily checked that ρ is an $A \widehat{\otimes} B$-bimodule morphism. Since $\pi_{A \widehat{\otimes} B} = (\pi_A \otimes \pi_B) \circ \theta^{-1}$, we have $\pi_{A \widehat{\otimes} B} \circ \rho = I_A \otimes I_B = I_{A \widehat{\otimes} B}$. Further $\|\rho\| \le C_1C_2$. Therefore $A \widehat{\otimes} B$ is C_1C_2-biprojective. □

Let A and B be Banach algebras, let $E \in A$-**mod**-A, and let $F \in B$-**mod**-B. We regard $E \widehat{\otimes} F$ as a Banach $A \widehat{\otimes} B$-bimodule with the multiplication given by

$$\left.\begin{aligned} (a \otimes b) \cdot (x \otimes y) &= (a \cdot x) \otimes (b \cdot y) \\ (x \otimes y) \cdot (a \otimes b) &= (x \cdot a) \otimes (y \cdot b) \end{aligned}\right\} \quad (a \in A,\ b \in B,\ x \in E, y \in F)\,.$$

We also regard $\mathcal{B}(E, F)$ as a Banach $A \widehat{\otimes} B$-bimodule with the multiplication given by

$$\left.\begin{aligned} ((a \otimes b) * T)(x) &= b \cdot T(x \cdot a) \\ (T * (a \otimes b))(x) &= T(a \cdot x) \cdot b \end{aligned}\right\} \quad (T \in \mathcal{B}(E, F),\ a \in A,\ b \in B,\ x \in E)\,.$$

We denote this module by $\widetilde{\mathcal{B}}(E,F)$. Similarly $\mathcal{B}(F,E)$ is a Banach $A\widehat{\otimes} B$-bimodule with the multiplication given by

$$\left.\begin{array}{l}((a\otimes b)*T)(x)=a\cdot T(x\cdot b)\\ (T*(a\otimes b))(x)=T(b\cdot x)\cdot a\end{array}\right\}\quad (T\in\mathcal{B}(F,E),\, a\in A,\, b\in B,\, x\in E)\,.$$

We denote this module by $\widehat{\mathcal{B}}(F,E)$. Let $\lambda\in(E\widehat{\otimes} F)'$. We define $\widetilde{T_\lambda}\in\mathcal{B}(E,F')$ and $\widehat{T_\lambda}\in\mathcal{B}(F,E')$ by

$$\left.\begin{array}{l}\left\langle y,\widetilde{T_\lambda}(x)\right\rangle=\langle x\otimes y,\lambda\rangle\\ \left\langle x,\widehat{T_\lambda}(y)\right\rangle=\langle x\otimes y,\lambda\rangle\end{array}\right\}\quad (x\in E,\, y\in F)\,.$$

Then the maps

$$\lambda\mapsto\widetilde{T_\lambda},\quad (E\widehat{\otimes} F)'\to\widetilde{\mathcal{B}}(E,F')\quad\text{and}\quad\lambda\mapsto\widehat{T_\lambda},\quad (E\widehat{\otimes} F)'\to\widehat{\mathcal{B}}(E,F')$$

are isometric $A\widehat{\otimes} B$-bimodule isomorphisms.

Proposition 2.3.9. *Let A be a C_1-biflat Banach algebra, and let B be a C_2-biflat Banach algebra. Then $A\widehat{\otimes} B$ is C_1C_2-biflat.*

Proof. There exist an A-bimodule morphism $\rho_A:(A\widehat{\otimes} A)'\to A'$ with $\rho_A\circ\pi_A'=I_A$ and $\|\rho_A\|\le C_1$ and a B-bimodule morphism $\rho_B:(B\widehat{\otimes} B)'\to B'$ with $\rho_B\circ\pi_B'=I_{B'}$ and $\|\rho_B\|\le C_2$.

Now let $\rho_{A\widehat{\otimes} B}$ denote map given by the composition

$$\begin{aligned}((A\widehat{\otimes} B)\widehat{\otimes}(A\widehat{\otimes} B))'\cong\widetilde{\mathcal{B}}(A\widehat{\otimes} A,(B\widehat{\otimes} B)')&\overset{T\mapsto\rho_B\circ T}{\longrightarrow}\widetilde{\mathcal{B}}(A\widehat{\otimes} A,B')\\ &\cong\widehat{\mathcal{B}}(B,(A\widehat{\otimes} A)')\overset{T\mapsto\rho_A\circ T}{\longrightarrow}\widehat{\mathcal{B}}(B,A')\cong(A\widehat{\otimes} B)'\,.\end{aligned}$$

It is immediately checked that each of the maps is an $A\widehat{\otimes} B$-bimodule morphism, and that $\left\|\rho_{A\widehat{\otimes} B}\right\|\le C_1C_2$. Take $\lambda\in(A\widehat{\otimes} B)'$. We follow $\pi'_{A\widehat{\otimes} B}(\lambda)$ under the sequence of compositions $\rho_{A\widehat{\otimes} B}$. We have

$$\pi'_{A\widehat{\otimes} B}(\lambda)\mapsto\pi_B'\circ\widetilde{T_\lambda}\circ\pi_A\mapsto\widetilde{T_\lambda}\circ\pi_A\mapsto\pi_A'\circ\widehat{T_\lambda}\mapsto\widehat{T_\lambda}\mapsto\lambda\,.$$

Hence $\rho_{A\widehat{\otimes} B}\circ\pi'_{A\widehat{\otimes} B}=I_{(A\widehat{\otimes} B)'}$, and therefore $A\widehat{\otimes} B$ is C_1C_2-biflat. □

We now prove a partial converse to Propositions 2.3.8 and 2.3.9. For a similar result about amenability, see [24, Proposition 3.5].

Proposition 2.3.10. *Let A be a unital Banach algebra, and let B be a Banach algebra containing a non-zero idempotent b_0. Suppose that $A\widehat{\otimes} B$ is C-biflat [C-biprojective]. Then A is $C\,\|b_0\|$-biflat [$C\,\|b_0\|$-biprojective].*

Proof. First suppose that $A\widehat{\otimes}B$ is C-biprojective. Then there exists an $A\widehat{\otimes}B$-bimodule morphism $\rho : A\widehat{\otimes}B \to (A\widehat{\otimes}B)\widehat{\otimes}(A\widehat{\otimes}B)$ with $\pi_{A\widehat{\otimes}B}\circ\rho = I_{A\widehat{\otimes}B}$ and $\|\rho\| \le C$. We regard $A\widehat{\otimes}B$ as an A-bimodule with the actions given by

$$a_1\cdot(a_2\otimes b) = a_1a_2\otimes b, \quad \text{and} \quad (a_2\otimes b)\cdot a_1 = a_2a_1\otimes b \quad (a_1,a_2\in A, b\in B)\,.$$

Then we have

$$\begin{aligned}\rho(a_1a_2\otimes b_0) &= \rho((a_1\otimes b_0)(a_2\otimes b_0)) = (a_1\otimes b_0)\cdot\rho(a_2\otimes b_0)\\ &= a_1\cdot(e_A\otimes b_0)\cdot\rho(a_2\otimes b_0) = a_1\cdot\rho(a_2\otimes b_0) \quad (a_1,a_2\in A)\,.\end{aligned}$$

Similarly we can show a right-module version of this equation. Hence we have

$$\rho(a_1a_2\otimes b_0) = a_1\cdot\rho(a_2\otimes b_0) = \rho(a_1\otimes b_0)\cdot a_2 \quad (a_1,a_2\in A)\,. \tag{2.3}$$

Take $\varphi\in(B')_{[1]}$ with $\langle b_0,\varphi\rangle = 1$ and define

$$\theta : (a_1\otimes b_1)\otimes(a_2\otimes b_2)\mapsto \varphi(b_1b_2)a_1\otimes a_2, \quad (A\widehat{\otimes}B)\widehat{\otimes}(A\widehat{\otimes}B)\to A\widehat{\otimes}A\,.$$

Then θ is an A-bimodule morphism.

We now define $\widetilde{\rho} : A\to A\widehat{\otimes}A$ by

$$\widetilde{\rho}(a) = \theta\circ\rho(a\otimes b_0) \quad (a\in A)\,.$$

By (2.3), $\widetilde{\rho}$ is an A-bimodule morphism. It follows from the identity

$$\pi_A\circ\theta = (I_A\otimes\varphi)\circ\pi_{A\widehat{\otimes}B}$$

that $\pi_A\circ\widetilde{\rho} = I_A$. Further $\|\widetilde{\rho}\| \le C\,\|b_0\|$. Therefore A is $C\,\|b_0\|$-biprojective.

The proof for biflatness is similar. Given an $A\widehat{\otimes}B$-bimodule morphism $\rho : A\widehat{\otimes}B \to ((A\widehat{\otimes}B)\widehat{\otimes}(A\widehat{\otimes}B))''$ with $\pi''_{A\widehat{\otimes}B}\circ\rho = \iota_{A\widehat{\otimes}B}$, we define $\widetilde{\rho} : A\to(A\widehat{\otimes}A)''$ by

$$\widetilde{\rho}(a) = \theta''\circ\rho(a\otimes b_0) \quad (a\in A)\,.$$

It is easily checked that $\widetilde{\rho}$ has the required properties. □

Proposition 2.3.11. *Let A be a unital Banach algebra, and let Λ be a non-empty set. Then $\mathbb{M}_\Lambda(A)$ is C-biflat [C-biprojective] if and only if A is C-biflat [C-biprojective].*

Proof. Set $B = \mathbb{M}_\Lambda(\mathbb{C})$. First note that B is 1-biprojective. Indeed, fix $k_0\in\Lambda$ and define $\rho : B\to B\widehat{\otimes}B$ by

$$\rho(a) = \sum_{i,j\in\Lambda} a_{ij}E_{ik_0}\otimes E_{k_0j} \quad (a = (a_{ij})\in B)\,.$$

The sum converges since $\sum_{i,j}|a_{ij}| < \infty$. It is easily checked on matrix units that ρ is a B-bimodule morphism which is a right inverse to the multiplication map.

Although B is not unital if the index set Λ is infinite, B always contains non-zero idempotents of norm 1. Since $\mathbb{M}_\Lambda(A) = A\widehat{\otimes}B$, the result follows from the preceding propositions. □

The analogous result for 'C-amenable' is not true. See [5, Theorem 2.7] for a relation between $AM(\mathbb{M}_\Lambda(A))$ and $AM(A)$.

Proposition 2.3.12. *Let A and B be unital Banach algebras, and let $E \in$ **mod**-A and $F \in$ **mod**-B be unital modules. Suppose that $E \widehat{\otimes} F$ is C-injective in **mod**-$A \widehat{\otimes} B$. Then E is C-injective in **mod**-A.*

Proof. Since E and F are unital, $E \widehat{\otimes} F$ is faithful in **mod**-$A \widehat{\otimes} B$. Hence there is a right $A \widehat{\otimes} B$-module morphism $\rho : \mathcal{B}(A \widehat{\otimes} B, E \widehat{\otimes} F) \to E \widehat{\otimes} F$ with $\rho \circ \Pi_{E \widehat{\otimes} F} = I_{E \widehat{\otimes} F}$. Take $x_0 \in F_{[1]}$ and $\mu \in F'_{[1]}$ with $\langle x_0, \mu \rangle = 1$. Consider the diagram

$$\begin{array}{ccc} \mathcal{B}(A \widehat{\otimes} B, E \widehat{\otimes} F) & \xrightarrow{\rho} & E \widehat{\otimes} F \\ {\scriptstyle T \mapsto \widetilde{T}} \uparrow & & \downarrow {\scriptstyle I_E \otimes \mu} \\ \mathcal{B}(A, E) & \xrightarrow{\widetilde{\rho}} & E \end{array},$$

where $\widetilde{T} = T \otimes \Pi_F(x_0)$ $(T \in \mathcal{B}(A, E))$, and set

$$\widetilde{\rho}(T) = (I_E \otimes \mu) \circ \rho(\widetilde{T}) \quad (T \in \mathcal{B}(A, E)).$$

We have

$$\widetilde{T \cdot a} = \widetilde{T} \cdot (a \otimes e_B) \quad (a \in A, T \in \mathcal{B}(A, E)),$$

and since F is unital we have

$$(I_E \otimes \mu)(z \cdot (a \otimes e_B)) = (I_E \otimes \mu)(z) \cdot a \quad (a \in A, z \in E \widehat{\otimes} F).$$

It follows that $\widetilde{\rho}$ is a right A-module morphism. We also have

$$\widetilde{\Pi_E(x)} = \Pi_E(x) \otimes \Pi_F(x_0) = \Pi_{E \widehat{\otimes} F}(x \otimes x_0) \quad (x \in E),$$

from which it follows that $\widetilde{\rho} \circ \Pi_E = I_E$. Therefore E is injective in **mod**-A.

The statement about the associated constants is clear. □

Chapter 3

Modules over measure algebras

Let G be a locally compact group. There are many standard left (and right) Banach $L^1(G)$ and $M(G)$-modules. In [8] the authors characterize the locally compact groups G such that various left Banach modules are, respectively, projective, injective, and flat over the group algebra $L^1(G)$. In this chapter we shall seek to obtain similar characterizations of modules over $M(G)$. We shall also improve slightly some of the results in [8] about $L^1(G)$-modules and give a simple answer to one of the open questions raised in [8]. Our conclusions are summarized in a table at the end of the chapter.

The $M(G)$-predual module $C_0(G)$ has received some attention before. It has been proved in [38] that the $M(G)$-module $C_0(G)$ is not injective for infinite discrete groups, and in [32] for general infinite locally compact groups.

3.1 Essential and faithful $L^1(G)$-modules

Theorem 3.1.1. *Let G be a locally compact group, and let $E \in M(G)$-**unmod**.*

(i) *Suppose that E is projective in $L^1(G)$-**mod**. Then E is projective in $M(G)$-**mod**.*

(ii) *Conversely, suppose that E is projective in $M(G)$-**mod** and that E is essential in $L^1(G)$-**mod**. Then E is projective in $L^1(G)$-**mod**.*

Proof. Set $I = L^1(G)$ and $A = M(G)$.

Suppose that E is projective in I-**mod**. Then there exists $\rho \in {}_I\mathcal{B}(E, I^\flat \widehat{\otimes} E)$ with $\pi_{I^\flat \widehat{\otimes} E} \circ \rho = I_E$. Let $Q : I^\flat \to A$ be the map $\alpha e^\flat + a \mapsto \alpha e_A + a$. Define $\widetilde{\rho} : E \to A \widehat{\otimes} E$ by $\widetilde{\rho} = (Q \otimes I_E) \circ \rho$. Then $\widetilde{\rho}$ is an I-module morphism. Since A is faithful in I-**mod** [4, 3.3.24], by Lemma 2.1.4 $A \widehat{\otimes} E$ is faithful in I-**mod**. Hence, by Lemma 2.3.2(ii), $\widetilde{\rho}$ is an A-module morphism. Since E is a unital $M(G)$-module we have $\pi_{A \widehat{\otimes} E} \circ (Q \otimes I_E) = \pi_{I^\flat \widehat{\otimes} E}$, and hence $\pi_{A \widehat{\otimes} E} \circ \widetilde{\rho} = I_E$. Therefore E is projective in A-**mod**.

The converse is Proposition 2.3.3. □

We cannot remove the condition that the module be essential in $L^1(G)$-**mod** in the implication 'E projective in $M(G)$-**mod** $\Rightarrow$ E projective in $L^1(G)$-**mod**'. A counter-example is the $M(G)$-module $M(G)$. This module is projective in $M(G)$-**mod** for all locally compact groups G since $M(G)$ is unital. However $M(G)$ is projective in $L^1(G)$-**mod** if and only if G is discrete [8, Theorem 2.6].

By Proposition 2.3.4 we have the following.

Theorem 3.1.2. *Let G be a locally compact group, and let $E \in$ **mod**-$M(G)$. Suppose that E is faithful in **mod**-$L^1(G)$. Then E is injective in **mod**-$L^1(G)$ if and only if E is injective in **mod**-$M(G)$.* □

Corollary 3.1.3. *Let G be an amenable locally compact group, and let $E \in M(G)$-**mod**. Suppose that E is essential in $L^1(G)$-**mod**. Then E is flat in $M(G)$-**mod**.*

Proof. By Johnson's theorem, the Banach algebra $L^1(G)$ is amenable. By Proposition 1.3.11, E is flat in $L^1(G)$-**mod**. Since E is essential the dual module E' is faithful. It now follows from Theorem 3.1.2 that E is flat in $M(G)$-**mod**. □

3.2 The module $M(G)$

The following confirms a conjecture of Dales and Polyakov [8, p. 403].

Theorem 3.2.1. *Let G be a locally compact group. Then $M(G)$ is flat in $L^1(G)$-**mod** for all G.*

Proof. Let $A = L^1(G)$ and $E = M(G)$. Then $\overline{AE} = A$, and this is certainly a flat A-module, so the result follows from Proposition 2.1.11. □

Corollary 3.2.2. *Let G be a locally compact group, and let $E \in M(G)$-**mod**. Suppose that E is flat in $M(G)$-**mod**. Then E is flat in $L^1(G)$-**mod**.*

Proof. We have shown that the inclusion $L^1(G) \to M(G)$ is a flat homomorphism. The result now follows from [40, Proposition 4.18]. □

3.3 The modules $L^\infty(G)$ and $L^1(G)''$

Set $I = L^1(G)$ and $E = L^\infty(G)$. We determine the essential part of E. Since I has a bounded approximate identity, this is $I \cdot E$. By [4, 3.3.13(i) and 3.3.23] $I \cdot E = LUC(G)$, the *bounded, left uniformly continuous functions on* G (thus $\lambda \in LUC(G)$ if and only if $\lambda \in C^b(G)$ and the map $t \mapsto t \cdot \lambda$, $G \to C^b(G)$ is continuous). Therefore E is essential if and only if G is discrete. Note that if G is compact, then $I \cdot E = LUC(G) = C(G)$.

Theorem 3.3.1. *Let G be a locally compact group. Then $L^\infty(G)$ is a projective $M(G)$-module if and only if G is finite.*

Proof. Set $E = L^\infty(G)$, $I = L^1(G)$, and $A = M(G)$. Certainly E is a projective A-module when G is finite.

For the converse, suppose that G is not compact. Then the second part of the proof of [8, Theorem 3.1] produces a contradiction. Thus G is compact. Now by Proposition 2.1.1 there is a projection of $L^\infty(G)$ onto $C(G)$. This is a contradiction of [8, Proposition 1.14] in the case where G is infinite. Hence G is finite. □

Theorem 3.3.2. *Let G be a locally compact group, and suppose that $L^1(G)''$ is projective in $M(G)$-**mod**. Then G is discrete and contains no infinite, amenable subgroup.*

Proof. Set $I = L^1(G)$. Since $M(G)^{\perp I} = 0$, [4, 3.3.24], it follows from Proposition 2.1.6 that $L^1(G)''^{\perp I} = 0$. This is the case only if G is discrete. That G contains no infinite amenable subgroup now follows from [8, Theorem 2.7]. □

3.4 Amenability and injectivity

Again let G be a locally compact group. Let us consider a right $L^1(G)$-module E which is the dual of a left $L^1(G)$-module. In the case where G is amenable, it follows from Proposition 1.3.11 that E is injective in **mod**-$L^1(G)$. In [8, §4] a converse to this result is proved under additional hypotheses. One of these hypotheses in [8, Theorem 4.6] is that the module be faithful in **mod**-$L^1(G)$. The result is then applied to the $L^1(G)$-module $L^1(G)''$. However this latter module is not faithful. The good news is that we can remove the hypothesis that the module be faithful in **mod**-$L^1(G)$, so that the conclusion is still valid.

Theorem 3.4.1. *Let G be a locally compact group, and let (E, λ, φ_G) be an augmentation-invariant Banach right $M(G)$-module. Suppose that E is a dual space, and that E is injective in **mod**-$M(G)$. Then G is amenable.*

Proof. Let E have a Banach space predual F. Set $A = M(G)$ and $I = L^1(G)$. Let $(v_\alpha) \subset I \widehat{\otimes} F$ be the net given by Theorem 2.2.12. Take $x \in E_{[1]}$ with $\langle x, \lambda \rangle = 1$. We have

$$\|v_\alpha\|_\pi \geq \|\widetilde{\pi}(v_\alpha)\|_{F''} \geq |\langle x, \widetilde{\pi}(v_\alpha) \rangle| \geq 1/2$$

for large enough α. Hence by passing to a subnet we may suppose that, for each α, $\|v_\alpha\|_\pi \geq 1/2$. We use the identification $I \widehat{\otimes} F = L^1(G, F)$ to define a net (k_α) in I by

$$k_\alpha(s) = \frac{\|v_\alpha(s)\|_F}{\|v_\alpha\|_\pi} \quad (s \in G)\,.$$

Then $k_\alpha \geq 0$, and

$$\|k_\alpha\|_1 = \int_G k_\alpha(s)\,\mathrm{d}m(s) = \int_G \frac{\|v_\alpha(s)\|_F}{\|v_\alpha\|_\pi}\,\mathrm{d}m(s) = 1\,.$$

Now take $t \in G$. We have

$$\|t \cdot k_\alpha - k_\alpha\|_1 \leq \frac{1}{\|v_\alpha\|_\pi} \int_G \left\|v_\alpha(t^{-1}s) - v_\alpha(s)\right\|_F \mathrm{d}m(s) \leq 2\,\|t \cdot v_\alpha - v_\alpha\|_\pi\,.$$

Hence $\lim_\alpha \|t \cdot k_\alpha - k_\alpha\|_1 = 0$. Therefore by Proposition 1.5.4, G is amenable. □

The same result holds if E is injective in **mod**-$L^1(G)$. This combines with Johnson's theorem to prove the following theorem.

Theorem 3.4.2. *Let G be a locally compact group, and let (E, λ, φ_G) be an augmentation-invariant Banach right $L^1(G)$-module. Suppose that E is a dual space. Then E is injective in* **mod**-$L^1(G)$ *if and only if G is amenable.* □

This is an improvement on [8, Theorem 4.6]. We have removed two hypotheses; that the module E be faithful in $L^1(G)$-**mod** and that E be a dual module. An analogous result holds if we exchange 'right' and 'left' in the above theorems.

Example 3.4.3. Let G be a locally compact group.

(i) The $L^1(G)$-modules $L^1(G)$ and $M(G)$ are augmentation-invariant. The augmentation character φ_G is an augmentation-invariant functional.

(ii) The module $L^\infty(G)$ is augmentation-invariant if and only if G is amenable.

(iii) $L^1(G)''$ is augmentation-invariant.

3.5 Summary

The following table summarizes our results about modules over the measure algebra. The table gives necessary and sufficient conditions for the specified Banach left $M(G)$-module in the first column to have the specified homological property in the top row. We take $1 < p < \infty$. The modules $L^1(G), C_0(G)$, and $L^p(G)$ are essential in $L^1(G)$-**mod**, and the modules $L^1(G), C_0(G), L^\infty(G), M(G)$, and $L^p(G)$ are faithful in $L^1(G)$-**mod**. Hence results about these modules follow by combining Theorems 3.1.1 and 3.1.2 and the results in [8]. The first number in each column refers to a theorem in this thesis. For the essential and faithful modules mentioned above the number in the second column refers to a theorem in [8], where the corresponding result is established for $L^1(G)$. These together establish the result for $M(G)$.

The indication ($\Rightarrow G$ amenable) means that the result implies that G is amenable. The indication (3) means that the result implies that G is discrete and contains no infinite, amenable subgroups. The indication ($\Rightarrow G$ super p-amenable) means that the result implies that G is super p-amenable (see Chapter 5).

Table 3.1: Modules over $M(G)$

	Projective	Injective	Flat
$L^1(G)$	all G **3.1.1** [4, 3.3.32]	G discrete and amenable **3.1.2**, 4.9	all G
$C_0(G)$	G compact **3.1.1**, 3.1	G finite **3.1.2**, 3.8	G amenable **3.1.2**, 4.7
$L^\infty(G)$	G finite **3.3.1**	all G	($\Rightarrow G$ amenable) **3.4.1**
$M(G)$	all G	G amenable **3.1.2**, 4.7	all G
$L^1(G)''$	(3) **3.3.2**	($\Rightarrow G$ amenable) **3.4.1**	**Not Known** *
$L^p(G)$	G compact **3.1.1**, 5.1	($\Rightarrow G$ super p-amenable)	($\Rightarrow G$ super q-amenable)

* Suppose that $L^1(G)''$ is flat in $M(G)$-**mod**. Then by Corollary 3.2.2 $L^1(G)''$ is flat in $L^1(G)$-**mod**. It is conjectured in [8] that this implies that G is amenable. If this conjecture is true, then $L^1(G)''$ is projective in $L^1(G)$-**mod** if and only if G is finite. We do not know if this latter result is true, and it seems quite a challenge.

Chapter 4

Multi-normed spaces

In this chapter we break from the main theme of the thesis and study the theory of multi-norms. These were introduced by Dales and Polyakov in [7]. The original motivation was to generalize the concept of an amenable locally compact group with the aim of answering questions about modules over $L^1(G)$. We shall take up this theme in the next chapter. Here we study the general theory of multi-norms. We introduce a generalization of the original concept. We define a *type-p multi-normed space* for any $1 \leq p \leq \infty$. The original definition in [7] corresponds to a type-∞ multi-normed space. This generality allows us to encompass *operator sequence spaces* introduced in [26]. Indeed, every operator sequence space is a type-2 multi-normed space.

4.1 Preliminaries

Let E be a linear space, and let $m, n \in \mathbb{N}$. The space $\mathbb{M}_{m,n}$ acts naturally as a map from E^n to E^m via

$$x \mapsto a \cdot x, \quad \mathbb{M}_{n,1}(E) \to \mathbb{M}_{m,1}(E) \quad (a \in \mathbb{M}_{m,n}) .$$

For $a \in \mathbb{M}_{m,n}$ we denote this linear operator by L_a. The dual of $L_a : E^n \to E^m$ is the operator $L_{a^T} : (E')^m \to (E')^n$, where $a^T \in \mathbb{M}_{n,m}$ denotes the transpose of a.

Let $1 \leq p \leq \infty$. For $a = (a_{ij}) \in \mathbb{M}_{m,n}$ we set

$$\|a\|_p = \|L_a\|_{\mathcal{B}(\ell_n^p, \ell_m^p)} .$$

We have

$$\|a\|_\infty = \max \left\{ \sum_{j=1}^{n} |a_{ij}| : i \in \mathbb{N}_m \right\} , \tag{4.1}$$

the maximum of the row sums, and

$$\|a\|_1 = \max \left\{ \sum_{i=1}^{m} |a_{ij}| : j \in \mathbb{N}_n \right\} , \tag{4.2}$$

the maximum of the column sums. For general $a \in \mathbb{M}_{m,n}$ and $1 < p < \infty$ there is no such formula for $\|a\|_p$.

For $a \in M_{m,n}$ and $k \in \mathbb{N}$ we set

$$a^{(k)} = \begin{pmatrix} a & & & \\ & a & & \\ & & \ddots & \\ & & & a \end{pmatrix} \in \mathbb{M}_{mk,nk}\,,$$

where there are k copies of a on the diagonal and 0's elsewhere. For each $1 \leq p \leq \infty$ we have

$$\left\|a^{(k)}\right\|_p = \|a\|_p\,.$$

A matrix $a \in \mathbb{M}_{m,n}$ is *row-special* [*column-special*] if a has at most 1 non-zero entry in each row [column]. If a has at most 1 non-zero entry in each row and each column then a is *special*.

Proposition 4.1.1. *Let $1 \leq p \leq \infty$ with conjugate index q, and let $m, n \in \mathbb{N}$.*

(i) *For each row-special matrix $a \in \mathbb{M}_{m,n}$ we have*

$$\|a\|_p = \max\left\{\left(\sum_{i=1}^{m} |a_{ij}|^p\right)^{1/p} : j \in \mathbb{N}_n\right\}\,,$$

which is the maximum of the p-norms of the columns of a.

(ii) *For each column-special matrix $a \in \mathbb{M}_{m,n}$ we have*

$$\|a\|_p = \max\left\{\left(\sum_{j=1}^{n} |a_{ij}|^q\right)^{1/q} : i \in \mathbb{N}_m\right\}\,,$$

which is the maximum of the q-norms of the rows of a.

Proof. (i) If $p = \infty$, then the result is easily deduced from (4.1). Suppose that $1 \leq p < \infty$. Let $a \in \mathbb{M}_{m,n}$ be a row-special matrix. Let the non-zero entry of a in row i be in column $j(i)$. This entry is $a_{i,j(i)}$. If there is no non-zero entry in row i then we set $a_{i,j(i)} = 0$. Take $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{C}^n$. We have

$$a \cdot \alpha = (a_{1,j(1)}\alpha_{j(1)}, \ldots, a_{m,j(m)}\alpha_{j(m)})\,.$$

Hence

$$\begin{aligned}\|a \cdot \alpha\|_p^p &= \sum_{i=1}^{m} \left|a_{i,j(i)}\right|^p \left|\alpha_{j(i)}\right|^p = \sum_{j=1}^{n} \left(\sum_{\{i \in \mathbb{N}_m : j(i) = j\}} |a_{i,j}|^p\right) |\alpha_j|^p \\ &= \sum_{j=1}^{n} \left(\sum_{i=1}^{m} |a_{i,j}|^p\right) |\alpha_j|^p \\ &\leq \max\left\{\sum_{i=1}^{m} |a_{ij}|^p : j \in \mathbb{N}_n\right\} \sum_{j=1}^{n} |\alpha_j|^p\,.\end{aligned}$$

From this it follows that $\|a\|_p \le \max\left\{(\sum_{i=1}^m |a_{ij}|^p)^{1/p} : j \in \mathbb{N}_n\right\}$.

To obtain the reverse inequality set $\alpha = (0,\ldots,1,\ldots,0)$ where 1 is in position j. Then we have $\|a\| \ge \|a\cdot\alpha\| = (\sum_{i=1}^m |a_{ij}|^p)^{1/p}$. This is true for each $j \in \mathbb{N}_n$, and so the result follows.

(ii) The transpose of a column-special matrix is row special. Hence this can be deduced from (i) with the observation

$$\|a\|_p = \|L_a\|_{\mathcal{B}(\ell_m^p,\ell_n^p)} = \|L_a'\|_{\mathcal{B}(\ell_n^q,\ell_m^q)} = \|L_{a^T}\|_{\mathcal{B}(\ell_n^q,\ell_m^q)} = \left\|a^T\right\|_q . \qquad \square$$

4.2 Special-normed spaces

Definition 4.2.1. Let $(E, \|\cdot\|)$ be a Banach space. A *special-norm* over E is a sequence $(\|\cdot\|_n : n \in \mathbb{N})$ such that $\|\cdot\|_n$ is a norm on E^n for each $n \in \mathbb{N}$, with $\|\cdot\|_1 = \|\cdot\|$ on $E = E^1$, and such that

$$\|a\cdot x\|_m \le \max_{i,j} |a_{i,j}|\, \|x\|_n \quad (x \in E^n) \tag{4.3}$$

for each $m,n \in \mathbb{N}$ and each special matrix $a \in \mathbb{M}_{m,n}$. The space E equipped with a special norm is a *special-normed space*.

Let $n \in \mathbb{N}$. We denote by $\mathfrak{S}_n$ the group of all permutations of the set $\mathbb{N}_n$. Let E be a linear space, let $\sigma \in \mathfrak{S}_n$, and let $\alpha = (\alpha_i) \in \mathbb{C}^n$. Then we define the operators $A_\sigma, M_\alpha : E^n \to E^n$ by

$$A_\sigma(x) = (x_{\sigma(1)},\ldots,x_{\sigma(n)}), \quad M_\alpha(x) = (\alpha_1 x_1,\ldots,\alpha_n x_n)\,,$$

where $x = (x_1,\ldots,x_n) \in E^n$.

Proposition 4.2.2. *Let E be a normed space, and let $(\|\cdot\|_n : n \in \mathbb{N})$ be a sequence such that $\|\cdot\|_n$ is a norm on E^n for each $n \in \mathbb{N}$, with $\|\cdot\|_1 = \|\cdot\|$ on $E = E^1$. Then the sequence $(\|\cdot\|_n : n \in \mathbb{N})$ is a special-norm if and only if for each $n \ge 2$ the following axioms hold:*

(A1) $\|A_\sigma(x)\|_n = \|x\|_n \quad (\sigma \in \mathfrak{S}_n,\ x \in E^n)$;

(A2) $\|M_\alpha(x)\|_n \le (\max_{i\in\mathbb{N}_n} |\alpha_i|)\, \|x\|_n \quad (\alpha = (\alpha_i) \in \mathbb{C}^n,\ x \in E^n)$;

(A3) $\|(x_1,\ldots,x_{n-1},0)\|_n = \|(x_1,\ldots,x_{n-1})\|_{n-1} \quad (x_1,\ldots,x_{n-1} \in E)$.

Proof. To see that a special-normed space satisfies the axioms we plug-in specific special matrices in (4.3). Fix $n \ge 2$. Axiom (A1) holds if and only if (4.3) holds for each permutation matrix $a \in \mathbb{M}_n$. Axiom (A2) holds if and only if (4.3) holds for each diagonal matrix $a \in \mathbb{M}_n$. Axiom (A3) holds if and only if (4.3) holds for all matrices a of the form

$$\begin{pmatrix} 1 & 0 & \dots & 0 \\ 0 & 1 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 1 \\ 0 & 0 & \dots & 0 \end{pmatrix} \in \mathbb{M}_{n,n-1} \quad \text{and} \quad \begin{pmatrix} 1 & 0 & \dots & 0 & 0 \\ 0 & 1 & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & 0 \\ 0 & 0 & \dots & 1 & 0 \end{pmatrix} \in \mathbb{M}_{n-1,n}\,.$$

Therefore a special-norm satisfies (A1), (A2) and (A3).

Conversely, suppose that the sequence $(\|\cdot\|_n : n \in \mathbb{N})$ satisfies (A1)-(A3). Take $m, n \in \mathbb{N}$, $x \in E^n$ and a special matrix $a = (a_{ij}) \in \mathbb{M}_{m,n}$. Let the non-zero entry in row i (if it exists) be in column $j(i)$. This entry is $a_{i,j(i)}$. If there is no non-zero entry in column j then we set $j(i) = -1$ and $a_{i,-1} = 0$. We also set $x_{-1} = 0$. With this notation, and using (A3), we have

$$\|a \cdot x\|_m = \left\|(a_{1,j(1)}x_{j(1)}, \dots, a_{m,j(m)}x_{j(m)})\right\|_m \leq \max_{i\in\mathbb{N}_n} |a_{i,j}| \left\|(x_{j(1)}, \dots, x_{j(m)})\right\|_m .$$

By considering the cases $m > n, m = n$ and $m < n$ separately we can show that

$$\left\|(x_{j(1)}, \dots, x_{j(m)})\right\|_m \leq \|x\|_n .$$

The result follows. ☐

Example 4.2.3. Let E be a Banach space, and let $1 \leq p \leq \infty$. Then it is immediately checked that the family $\{\ell^p_n(E) : n \in \mathbb{N}\}$ satisfies axioms (A1)-(A3), and hence is a special-normed space.

We give some elementary properties of these spaces. For the next five lemmas we suppose that E is a special-normed space.

Lemma 4.2.4. *Let $n \in \mathbb{N}$, and let $x_1, \dots, x_n \in E$. Then*

$$\max_{i\in\mathbb{N}_n} \|x_i\| \leq \|(x_1, \dots, x_n)\|_n \leq \sum_{i=1}^{n} \|x_i\| .$$

Proof. For each $i \in \mathbb{N}_n$, by (A2) and (A3), we have

$$\|x_i\| = \|(0, \dots, 0, x_i, 0, \dots, 0)\|_n \leq \|(x_1, \dots, x_n)\|_n .$$

Hence $\max_{i\in\mathbb{N}_n} \|x_i\| \leq \|(x_1, \dots, x_n)\|_n$.

Now, by (A3) we have

$$\|(x_1, \dots, x_n)\|_n \leq \sum_{i=1}^{n} \|(0, \dots, 0, x_i, 0, \dots, 0)\|_n = \sum_{i=1}^{n} \|x_i\| .$$ ☐

Lemma 4.2.5. *For each $n \in \mathbb{N}$, $(E^n, \|\cdot\|_n)$ is a Banach space.*

Proof. By Lemma 4.2.4, for each $n \in \mathbb{N}$ and $x_1, \ldots, x_n \in E$, we have

$$\max_{i \in \mathbb{N}_n} \|x_i\| \leq \|(x_1, \ldots, x_n)\|_n \leq n \max_{i \in \mathbb{N}_n} \|x_i\| \ .$$

It follows that $(E^n, \|\cdot\|_n)$ is complete. □

Lemma 4.2.6. *Let $n \geq 2$, and let $x_1, \ldots, x_n \in E$. Then*

$$\|(x_1, \ldots, x_{n-1})\|_{n-1} \leq \|(x_1, \ldots, x_{n-1}, x_n)\|_n \ .$$

Proof. By (A2) and (A3), we have

$$\|(x_1, \ldots, x_{n-1})\|_{n-1} = \|(x_1, \ldots, x_{n-1}, 0)\|_{n-1} \leq \|(x_1, \ldots, x_{n-1}, x_n)\|_n \ . \qquad \square$$

Lemma 4.2.7. *Let $n \geq 2$, let $x_1, \ldots, x_n \in E$, and let $0 \leq t \leq 1$. Then*

$$\|(x_1, \ldots, x_{n-2}, tx_{n-1} + (1-t)x_n, tx_n + (1-t)x_{n-1})\|_n \leq \|(x_1, \ldots, x_{n-2}, x_{n-1}, x_n)\|_n \ .$$

Proof. Set $x = (x_1, \ldots, x_{n-2}, tx_{n-1} + (1-t)x_n, tx_n + (1-t)x_{n-1})$. The result follows from the identity

$$x = t(x_1, \ldots, x_{n-2}, x_{n-1}, x_n) + (1-t)(x_1, \ldots, x_{n-2}, x_n, x_{n-1}) \, .$$

Hence

$$\begin{aligned} \|x\|_n &\leq t \|(x_1, \ldots, x_{n-2}, x_{n-1}, x_n)\|_n + (1-t) \|(x_1, \ldots, x_{n-2}, x_n, x_{n-1})\|_n \\ &= \|(x_1, \ldots, x_{n-2}, x_{n-1}, x_n)\|_n \quad \text{by (A1)} \, . \end{aligned}$$

□

Lemma 4.2.8. *Let $n \geq 2$, and let $x_1, \ldots, x_{n-1} \in E$. Then*

(i) $\|(x_1, \ldots, x_{n-2}, x_{n-1}, x_{n-1})\|_n \leq \|(x_1, \ldots, x_{n-2}, 2x_{n-1})\|_{n-1}$.

(ii) $\|(x_1, \ldots, x_{n-2}, x_{n-1}, x_{n-1})\|_n \geq \|(x_1, \ldots, x_{n-2}, x_{n-1})\|_{n-1}$.

Proof. (i) We have

$$\begin{aligned} \|(x_1, \ldots, x_{n-2}, x_{n-1}, x_{n-1})\|_n &= \left\| \left(x_1, \ldots, x_{n-2}, \frac{2x_{n-1} + 0}{2}, \frac{0 + 2x_{n-1}}{2} \right) \right\|_n \\ &\leq \|(x_1, \ldots, x_{n-2}, 2x_{n-1}, 0)\|_n \quad \text{by Lemma 4.2.7} \\ &= \|(x_1, \ldots, x_{n-2}, 2x_{n-1})\|_{n-1} \quad \text{by (A3)} \, . \end{aligned}$$

(ii) This is a special case of Lemma 4.2.6 □

For us the most important classes of special-normed spaces are those where equality holds in either (i) or (ii) above. These are the focus of Section 4.6.

4.3 Type-p multi-normed spaces

Definition 4.3.1. Let $(E, \|\cdot\|)$ be a Banach space, and let $1 \leq p \leq \infty$. A *type-p multi-norm* over E is a sequence $(\|\cdot\|_n : n \in \mathbb{N})$ such that $\|\cdot\|_n$ is a norm on E^n for each $n \in \mathbb{N}$, with $\|\cdot\|_1 = \|\cdot\|$ on $E = E^1$, and such that

$$\|a \cdot x\|_m \leq \|a\|_p \|x\|_n \quad (a \in \mathbb{M}_{m,n},\, x \in E^n) \tag{4.4}$$

for each $m, n \in \mathbb{N}$. The space E equipped with a type-p multi-norm $(\|\cdot\|_n : n \in \mathbb{N})$ is a *type-p multi-normed space.*

Let $1 \leq p \leq \infty$, and let $a \in \mathbb{M}_{m,n}$ be a special matrix. Then

$$\|a\|_p = \sup_{i,j} |a_{i,j}| \,.$$

Hence a type-p multi-normed space is a special-normed space.

We now give some elementary properties of these spaces. Throughout the next lemmas we suppose that $1 \leq p \leq \infty$ with conjugate index q, and that E is a type-p multi-normed space.

Lemma 4.3.2. *Let $n \geq 2$, let $x_1, \ldots, x_{n-1} \in E$, and let $\alpha, \beta \in \mathbb{C}$. Then*

$$\|(x_1, \ldots, x_{n-2}, \alpha x_{n-1}, \beta x_{n-1})\|_n = \left\|(x_1, \ldots, x_{n-2}, \|(\alpha, \beta)\|_p\, x_{n-1})\right\|_{n-1} .$$

Proof. Set $\gamma = \|(\alpha, \beta)\|_p$, and set

$$x = (x_1, \ldots, x_{n-2}, \alpha x_{n-1}, \beta x_{n-1}), \quad y = (x_1, \ldots, x_{n-2}, \gamma x_{n-1})\,.$$

There exist $\lambda, \mu \in \mathbb{C}$ with

$$\|(\lambda, \mu)\|_q \leq 1 \quad \text{and} \quad \lambda\alpha + \mu\beta = \gamma\,.$$

Consider the matrices

$$a = \begin{pmatrix} 1 & 0 & \ldots & 0 & 0 & 0 \\ 0 & 1 & \ldots & 0 & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & \ldots & 1 & 0 & 0 \\ 0 & 0 & \ldots & 0 & \lambda & \mu \end{pmatrix} \in \mathbb{M}_{n-1,n}\,, \quad b = \begin{pmatrix} 1 & 0 & \ldots & 0 & 0 \\ 0 & 1 & \ldots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \ldots & 1 & 0 \\ 0 & 0 & \ldots & 0 & \lambda/\gamma \\ 0 & 0 & \ldots & 0 & \mu/\gamma \end{pmatrix} \in \mathbb{M}_{n,n-1} \,.$$

By Proposition 4.1.1 we have $\|a\|_p = \|b\|_p = 1$. By (4.4) we have

$$\|y\|_{n-1} = \|a \cdot x\|_{n-1} \leq \|x\|_n \quad \text{and}$$
$$\|x\|_n = \|b \cdot y\|_n \leq \|y\|_{n-1} \,.$$

Therefore $\|x\|_n = \|y\|_{n-1}$, as required. □

Lemma 4.3.3. *Let $n \geq 2$, and let $x_1, \ldots, x_{n-1} \in E$. Then*

$$\|(x_1, \ldots, x_{n-1}, x_{n-1})\|_n = \left\|(x_1, \ldots, 2^{1/p}x_{n-1})\right\|_{n-1} .$$

Proof. This is a special case of Lemma 4.3.2. □

Lemma 4.3.4. *Let $k \leq n \in \mathbb{N}$, let $x_1, \ldots, x_n \in E$, and $\alpha_1, \ldots, \alpha_k \in \mathbb{C}$. Set $\gamma = \|(\alpha_1, \ldots, \alpha_k)\|_q$. Then*

$$\left\|\left(x_1, \ldots, x_{n-k}, \sum_{i=1}^{k} \alpha_i x_{n-k+i}\right)\right\|_{n-k+1} \leq \|(x_1, \ldots, x_{n-k}, \gamma x_{n-k+1}, \ldots, \gamma x_n)\|_n .$$

Proof. Consider the matrix

$$a = \begin{pmatrix} 1 & 0 & \ldots & 0 & 0 & \ldots & 0 \\ 0 & 1 & \ldots & 0 & 0 & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & 0 \\ 0 & 0 & \ldots & 1 & 0 & \ldots & 0 \\ 0 & 0 & \ldots & 0 & \alpha_1/\gamma & \ldots & \alpha_k/\gamma \end{pmatrix} \in \mathbb{M}_{n-k+1,n} .$$

We have

$$a \cdot (x_1, \ldots, x_{n-k}, \gamma x_{n-k+1}, \ldots, \gamma x_n) = \left(x_1, \ldots, x_{n-k}, \sum_{i=1}^{k} \alpha_i x_{n-k+i}\right)$$

and by Proposition 4.1.1 $\|a\|_p = 1$. Hence the result follows from (4.4). □

The following lemma also works for special-normed spaces.

Lemma 4.3.5 (Some standard constructions)**.** *Let E be a type-p multi-normed space.*

(i) *Let F be a closed subspace of E. Then F becomes a type-p multi-normed space via the linear embedding $F^n \to E^n$ $(n \in \mathbb{N})$.*

(ii) *Let $N \in \mathbb{N}$. Then E^N becomes a type-p multi-normed space via the identification $(E^N)^n = E^{nN}$ $(n \in \mathbb{N})$.*

Proof. (i) This is immediate from the definition.

(ii) Take $y = (y_1, \ldots, y_n) \in (E^N)^n$, say $y_i = (x_{i,1}, \ldots, x_{i,N})$ $(i \in \mathbb{N}_n)$. Take

$a \in \mathbb{M}_{m,n}$. Then we have

$$\begin{aligned}
\|a \cdot y\|_m &= \left\| \left(\sum_{i=1}^n a_{1,i} y_i, \dots, \sum_{i=1}^n a_{m,i} y_i \right) \right\|_m \\
&= \left\| \left(\sum_{i=1}^n a_{1,i} x_{i,1}, \dots, \sum_{i=1}^n a_{1,i} x_{i,N}, \dots, \sum_{i=1}^n a_{m,i} x_{i,1}, \dots, \sum_{i=1}^n a_{m,i} x_{i,N} \right) \right\|_{mN} \\
&= \left\| \left(\sum_{i=1}^n a_{1,i} x_{i,1}, \dots, \sum_{i=1}^n a_{m,i} x_{i,1}, \dots, \sum_{i=1}^n a_{1,i} x_{i,N}, \dots, \sum_{i=1}^n a_{m,i} x_{i,N} \right) \right\|_{mN} \quad \text{by (A1)} \\
&= \left\| a^{(N)} \cdot (x_{1,1}, \dots, x_{n,1}, \dots, x_{1,N}, \dots, x_{n,N}) \right\|_{mN} \\
&\le \|a\|_p \left\| (x_{1,1}, \dots, x_{1,N}, \dots, x_{n,1}, \dots, x_{n,N}) \right\|_{nN} \quad \text{by (A1)} \\
&= \|a\|_p \|y\|_n \, .
\end{aligned}$$

The result follows. □

Lemma 4.3.6. *Let E be a Banach space, and let $(\|\cdot\|_n : n \in \mathbb{N})$ be a sequence of norms on the spaces E^n $(n \in \mathbb{N})$ satisfying axiom* (A3). *Then the sequence $(\|\cdot\|_n : n \in \mathbb{N})$ is a type-p multi-norm if and only if*

$$\|a \cdot x\|_n \le \|a\|_p \|x\|_n \quad (a \in \mathbb{M}_n, \, x \in E^n)$$

for each $n \in \mathbb{N}$.

Proof. Take $n, m \in \mathbb{N}$, $x \in E^n$ and $a \in \mathbb{M}_{m,n}$. Set $N = \max\{n, m\}$. We define $\widetilde{a} \in \mathbb{M}_N$ by either adjoining $N - m$ rows to the bottom of a or $N - n$ columns to the right of a. We set $\widetilde{x} = (x_1, \dots, x_n, 0, \dots, 0) \in E^N$ where we have adjoined $N - n$ 0's to x. By hypothesis we have $\|\widetilde{a} \cdot \widetilde{x}\|_N \le \|\widetilde{a}\|_p \|\widetilde{x}\|_N = \|a\|_p \|\widetilde{x}\|_N$. By axiom (A3) we have $\|\widetilde{a} \cdot \widetilde{x}\|_N = \|a \cdot x\|_m$ and $\|\widetilde{x}\|_N = \|x\|_n$. The result follows. □

Lemma 4.3.7. *Let $1 \le p \le \infty$. Then the family $\{\ell_n^p : n \in \mathbb{N}\}$ is the unique type-p multi-normed space over $\mathbb{C}$.*

Proof. It is immediate from the definition that $\{\ell_n^p : n \in \mathbb{N}\}$ is a type-p multi-normed space. The uniqueness follows from Lemma 4.3.2. □

Example 4.3.8. Let E be a Banach space. Let $p = 1$ or $p = \infty$. Then it is easily checked using equations (4.1) and (4.2) that the family $\{\ell_n^p(E) : n \in \mathbb{N}\}$ is a type-p multi-normed space.

Now let $1 < p < \infty$. It follows from [25, §4] that the family $\{\ell_n^p(E) : n \in \mathbb{N}\}$ is a type-p multi-normed space if and only if E is isometrically isomorphic to a subspace of a quotient of an L^p-space. A more general result is proved in [27, Theorem 3.2].

4.4 Multi-bounded linear operators

Let E and F be linear spaces, let $T : E \to F$ be a linear mapping, and let $k \in \mathbb{N}$. Then we define the k^{th}-*amplification* of T, $T^{(k)} : E^k \to F^k$ by

$$T^{(k)}(x_1, \ldots, x_k) = (T(x_1), \ldots, T(x_k)) \quad ((x_1, \ldots, x_k) \in E^k) .$$

Definition 4.4.1. Let E and F be special-normed spaces, and let $T \in \mathcal{B}(E, F)$. Then T is *multi-bounded* if

$$\|T\|_{mb} = \sup_{k\in\mathbb{N}} \left\|T^{(k)}\right\|_{\mathcal{B}(E^k,F^k)} < \infty .$$

We set $\mathcal{M}(E, F) = \{T \in \mathcal{B}(E, F) : \|T\|_{mb} < \infty\}$. Then $\|\cdot\|_{mb}$ is a norm on $\mathcal{M}(E, F)$ called the *multi-bounded norm* and $\mathcal{M}(E, F)$ is the space of *multi-bounded operators.*

It is easy to check that $\mathcal{M}(E, F)$ is a Banach space. This definition of the multi-bounded norm agrees with that given in [7, Definition 5.11], and in the case where E and F are operator sequence spaces, it is the same as the *sequentially-bounded norm*. In this case, the multi-bounded operators are the same as the *sequentially bounded operators*. In the notation of [26] we have $\mathcal{M}(E, F) = \mathcal{SB}(E, F)$.

Example 4.4.2. Let E be a Banach space, and let $1 \le p < \infty$. Following the notation of [22] we define the *weak p-summing norm* on E^n by

$$\mu_{p,n}(x) = \sup \left\{ \left(\sum_{i=1}^{n} |\langle x_i, \lambda\rangle|^p \right)^{1/p} : \lambda \in E'_{[1]} \right\}$$

where $x = (x_1, \ldots, x_n) \in E^n$. These norms are denoted by $\|\cdot\|_p^{weak}$ in [9, p. 32]. It is clear that the family $(\mu_{p,n} : n \in \mathbb{N})$ is a special-norm over E.

Let E and F be Banach spaces, and let $1 \le p < \infty$. We regard E as a special-normed space equipped with the family of weak-p summing norms, and we regard F as a special-normed space by setting $F^n = \ell^p_n(F)$ $(n \in \mathbb{N})$. Then the multi-bounded operators from E to F are the same as the *p-summing operators* from E to F. In the notation of [33] we have $\mathcal{M}(E, F) = \mathcal{P}_p(E, F)$, and $\|T\|_{mb} = \pi_p(T)$ $(T \in \mathcal{M}(E, F))$.

Example 4.4.3. Let $p_1, p_2 \in [1, \infty]$. Let E be a type-p_1 multi-normed space, and let F be a type-p_2 multi-normed space. Consider the space $\mathcal{M}(E, F)$.

Case 1 $p_1 \le p_2$. Take $\lambda \in E'$ and $y \in F$, and set $T = y \otimes \lambda$. For each $k \in \mathbb{N}$

and $x_1, \ldots, x_k \in E$, we have

$$\begin{aligned}\|(T(x_1), \ldots, T(x_k))\|_k &= \|(\langle x_1, \lambda\rangle y, \ldots, \langle x_k, \lambda\rangle y)\|_k \\ &= \left(\sum_{i=1}^{k} |\langle x_i, \lambda\rangle|^{p_2}\right)^{1/p_2} \|y\| \quad \text{(by Lemma 4.3.2)} \\ &\leq \mu_{p_1,k}(x_1, \ldots, x_k) \|y\| \|\lambda\| \\ &\leq \|(x_1, \ldots, x_k)\|_k \|y\| \|\lambda\| \quad \text{(anticipating Prop 4.5.3)}\,.\end{aligned}$$

Hence $y \otimes \lambda \in \mathcal{M}(E, F)$ with $\|y \otimes \lambda\|_{mb} = \|y\| \|\lambda\|$. It follows that $\mathcal{F}(E, F) \subset \mathcal{M}(E, F)$.

Case 1 $p_1 > p_2$. Assume towards a contradiction that there exists $T \in \mathcal{M}(E, F)$ and $T \neq 0$. Take $x \in E \setminus \{0\}$ with $T(x) \neq 0$. Then, for each $k \in \mathbb{N}$, we have

$$\|T\|_{mb} \geq \left\|T^{(k)}\right\| \geq \frac{\|(T(x), \ldots, T(x))\|_k}{\|(x, \ldots, x)\|_k} = \frac{\|T(x)\|}{\|x\|} k^{1/p_2 - 1/p_1}\,.$$

Since $1/p_2 - 1/p_1 > 0$, this is a contradiction. Therefore $\mathcal{M}(E, F) = \{0\}$.

We now consider how to define a special-normed structure over $\mathcal{M}(E, F)$. Let E and F be linear spaces, and take $n \in \mathbb{N}$. For $T_1, \ldots, T_n \in \mathcal{L}(E, F)$ we define $\prod_{i=1}^{n} T_i \in \mathcal{L}(E, F^n)$ by

$$\left(\prod_{i=1}^{n} T_i\right)(x) = (T_1(x), \ldots, T_n(x)) \quad (x \in E)\,.$$

We also define $\bigoplus_{i=1}^{n} T_i \in \mathcal{L}(E^n, F)$ by

$$\left(\bigoplus_{i=1}^{n} T_i\right)(x_1, \ldots, x_n) = T_1(x_1) + \cdots + T_n(x_n) \quad (x_1, \ldots, x_n \in E)\,.$$

These two constructions are dual to each other in the following sense

$$\left(\prod T_i\right)' = \bigoplus T_i' \in \mathcal{L}((F')^n, E')$$

and

$$\left(\bigoplus T_i\right)' = \prod T_i' \in \mathcal{L}(F', (E')^n)\,.$$

Let E be a linear space, let $n \in \mathbb{N}$, and let $\|\cdot\|_n$ be a norm on E^n. Consider the following two conditions on $\|\cdot\|_n$:

(C1) $\|(0, \ldots, x_i, \ldots, 0)\|_n = \|x_i\| \quad (x_i \in E, i \in \mathbb{N}_n)$;

(C2) $\|(x_1, \ldots, x_n)\|_n \geq \max\{\|x_i\| : i \in \mathbb{N}_n\} \quad (x_1, \ldots, x_n \in E)$.

Proposition 4.4.4. *Let E and F be normed spaces. Let $n \in \mathbb{N}$, and let $\|\cdot\|_n$ be a norm on F^n satisfying* (C1) *and* (C2). *Then there is a linear bijection*

$$(T_1, \ldots, T_n) \mapsto \prod T_i, \quad \mathcal{B}(E, F)^n \to \mathcal{B}(E, F^n)\,.$$

Proof. Take $T_1, \ldots, T_n \in \mathcal{B}(E,F)$. For $x \in E$ we have

$$\left\| \left(\prod T_i \right)(x) \right\|_n \leq \sum_{i=1}^n \|(0, \ldots, T_i(x), \ldots, 0)\|_n \leq \left(\sum_{i=1}^n \|T_i\| \right) \|x\|$$

and so $\prod T_i \in \mathcal{B}(E, F^n)$ with $\|\prod T_i\| \leq \sum_{i=1}^n \|T_i\|$.

Let $\pi_i : F^n \to F$ be the projection onto the i^{th} coordinate. Take $T \in \mathcal{B}(E, F^n)$. For each $i \in \mathbb{N}_n$ and $x \in E$ we have

$$\|(\pi_i \circ T)(x)\| \leq \|T(x)\|_n \leq \|T\| \, \|x\|$$

and so $\pi_i \circ T \in \mathcal{B}(E,F)$ and $T = \prod(\pi_i \circ T)$. Hence the map is a surjection, it is clearly also an injection. □

Proposition 4.4.5. *Let E and F be normed spaces. Let $n \in \mathbb{N}$, and let $\|\cdot\|_n$ be a norm on E^n satisfying* (C1) *and* (C2)*. Then there is a linear bijection*

$$(T_1, \ldots, T_n) \mapsto \bigoplus T_i, \quad \mathcal{B}(E,F)^n \to \mathcal{B}(E^n, F)\,.$$

Proof. Take $T_1, \ldots, T_n \in \mathcal{B}(E,F)$. For $(x_1, \ldots, x_n) \in E^n$ we have

$$\begin{aligned} \left\| \left(\bigoplus T_i \right)(x_1, \ldots, x_n) \right\| &\leq \left(\sum_{i=1}^n \|T_i\| \right) \max\{\|x_i\| : i \in \mathbb{N}_n\} \\ &\leq \left(\sum_{i=1}^n \|T_i\| \right) \|(x_1, \ldots, x_n)\|_n\,, \end{aligned}$$

and so $\bigoplus T_i \in \mathcal{B}(E, F^n)$ with $\|\bigoplus T_i\| \leq \sum_{i=1}^n \|T_i\|$.

Let $\nu_i : E \to E^n$ be the embedding into the i^{th} coordinate. Take $T \in \mathcal{B}(E^n, F)$. For each $i \in \mathbb{N}_n$ and $x \in E$ we have

$$\|(T \circ \nu_i)(x)\| \leq \|T\| \, \|(0, \ldots, x, \ldots, 0)\|_n = \|T\| \, \|x\|\,,$$

and so $T \circ \nu_i \in \mathcal{B}(E,F)$ and $T = \bigoplus(T \circ \nu_i)$. Hence the map is a surjection, it is clearly also an injection. □

Special-normed spaces certainly satisfy the weak conditions (C1) and (C1), and by similar arguments we can show the following.

Proposition 4.4.6. *Let E and F be special-normed spaces. Then for each $n \in \mathbb{N}$, there are linear bijections*

$$(T_1, \ldots, T_n) \mapsto \prod T_i, \quad \mathcal{M}(E,F)^n \to \mathcal{M}(E, F^n)\,, \tag{4.5}$$

and

$$(T_1, \ldots, T_n) \mapsto \bigoplus T_i, \quad \mathcal{M}(E,F)^n \to \mathcal{M}(E^n, F)\,. \tag{4.6}$$

□

These identifications of linear spaces give us *two* possible ways of norming the linear space $\mathcal{M}(E,F)^n$ and hence defining a special-norm structure over $\mathcal{M}(E,F)$. We shall denote by $\|\cdot\|_n^\dagger$ the norm on $\mathcal{M}(E,F)^n$ induced by the identification (4.5), and denote by $\mathcal{M}(E,F)^\dagger$ the Banach space $\mathcal{M}(E,F)$ equipped with the family of norms $(\|\cdot\|_n^\dagger : n\in\mathbb{N})$. Similarly, we denote by $\|\cdot\|_n^\times$ the norm on $\mathcal{M}(E,F)^n$ induced by the identification (4.6), and denote by $\mathcal{M}(E,F)^\times$ the Banach space $\mathcal{M}(E,F)$ equipped with the family of norms $(\|\cdot\|_n^\times : n\in\mathbb{N})$. It is immediately checked that $\mathcal{M}(E,F)^\dagger$ and $\mathcal{M}(E,F)^\times$ are special-normed spaces.

Proposition 4.4.7. *Let $1\le p\le\infty$. Let E be a special-normed space, and let F be a type-p multi-normed space. Then $\mathcal{M}(E,F)^\dagger$ is a type-p multi-normed space.*

Proof. For each $n\in\mathbb{N}$, we set

$$\theta_n : (T_1,\dots,T_n)\mapsto \prod T_i\,,\quad \mathcal{M}(E,F)^n\to\mathcal{M}(E,F^n)\,.$$

Take $m,n\in\mathbb{N}$, $a\in\mathbb{M}_{m,n}$, and $T=(T_1,\dots,T_n)\in\mathcal{M}(E,F)^n$. We have

$$\theta_m(a\cdot T)^{(k)}(x)=a^{(k)}\cdot\left(\theta_n(T)^{(k)}(x)\right)\quad (x\in E^k,\ k\in\mathbb{N})\,,$$

from which it follows that

$$\left\|\theta_m(a\cdot T)^{(k)}\right\|_{\mathcal{B}(E^k,F^{nk})}\le\|a\|_p\left\|\theta_n(T)^{(k)}\right\|_{\mathcal{B}(E^k,F^{nk})}\quad (k\in\mathbb{N})\,.$$

Now we have

$$\|a\cdot T\|_m^\dagger=\|\theta_m(a\cdot T)\|_{mb}\le\|a\|_p\,\|\theta_n(T)\|_{mb}=\|a\|_p\,\|T\|_n^\dagger\,.$$ □

Proposition 4.4.8. *Let $1\le p\le\infty$ with conjugate index q. Let E be a type-p multi-normed space, and let F be a special-normed space. Then $\mathcal{M}(E,F)^\times$ is a type-q multi-normed space.*

Proof. For each $n\in\mathbb{N}$, we set

$$\sigma_n : (T_1,\dots,T_n)\mapsto \bigoplus T_i\,,\quad \mathcal{M}(E,F)^n\to\mathcal{M}(E^n,F)\,.$$

Take $m,n\in\mathbb{N}$, $a\in\mathbb{M}_{m,n}$, and $T=(T_1,\dots,T_n)\in\mathcal{M}(E,F)^n$. Set $b=a^T\in\mathbb{M}_{n,m}$. We have

$$\sigma_m(a\cdot T)^{(k)}(x)=\sigma_n(T)^{(k)}(b^{(k)}\cdot x)\quad (x\in E^{mk},\ k\in\mathbb{N})\,,$$

from which it follows that

$$\left\|\sigma_m(a\cdot T)^{(k)}\right\|_{\mathcal{B}(E^{mk},F^k)}\le\|a\|_q\left\|\sigma_n(T)^{(k)}\right\|_{\mathcal{B}(E^{mk},F^k)}\quad (k\in\mathbb{N})\,.$$

Now we have

$$\|a\cdot T\|_n^\times=\|\sigma_m(a\cdot T)\|_{mb}\le\|a\|_q\,\|\sigma_n(T)\|_{mb}=\|a\|_q\,\|T\|_n^\times\,.$$ □

In the case where E and F are type-∞ multi-normed spaces $\mathcal{M}(E,F)^\dagger$ agrees with the *multi-bounded* multi-norm structure on $\mathcal{M}(E,F)$ defined in [7, Definition 6.18]. This is also the same way that $\mathcal{SB}(E,F)$ is turned in to an operator sequence space in [26].

For a Banach space E and a special-normed space F we can define a special-normed structure over $\mathcal{B}(E,F)$ via (4.5). We denote this special-normed space by $\mathcal{B}(E,F)^\dagger$. Similarly for a special-normed space E and a Banach space F we can define a special-normed structure over $\mathcal{B}(E,F)$ via (4.6). We denote this special-normed space by $\mathcal{B}(E,F)^\times$.

Similar (simpler) arguments as above show the following.

Proposition 4.4.9. *Let $1 \le p \le \infty$. Let E be a Banach space, and let F be a type-p multi-normed space. Then $\mathcal{B}(E,F)^\dagger$ is a type-p multi-normed space.* □

Proposition 4.4.10. *Let $1 \le p \le \infty$ with conjugate index q. Let E be a type-p multi-normed space, and let F be a Banach space. Then $\mathcal{B}(E,F)^\times$ is a type-q multi-normed space.* □

Definition 4.4.11. Let E be a special-normed space. Then we set $E^\times = \mathcal{B}(E,\mathbb{C})^\times$.

Thus $E^\times$ is a special normed space whose underlying Banach space is E'. The level-n norm $\|\cdot\|_n^\times$ on $(E')^n$ is simply the dual of the norm $\|\cdot\|_n$ on E^n. The following is an immediate consequence of Proposition 4.4.10.

Corollary 4.4.12. *Let $1 \le p \le \infty$ with conjugate index q. Let E be a type-p multi-normed space. Then $E^\times$ is a type-q multi-normed space.* □

4.5 The maximum and minimum multi-norms

Let E be a Banach space, and take $p \in [1,\infty]$. Let $\{(\|\cdot\|_n^\alpha : n \in \mathbb{N}) : \alpha \in \Lambda\}$ be the (non-empty) set of type-p multi-norms over E. Then the *maximum* type-p multi-norm over E is defined by

$$\|x\|_n^{\max\text{-}p} = \sup_{\alpha\in\Lambda} \|x\|_n^\alpha \quad (x \in E^n,\, n \in \mathbb{N})\,.$$

The *minimum* type-p multi-norm over E is defined by

$$\|x\|_n^{\min\text{-}p} = \inf_{\alpha\in\Lambda} \|x\|_n^\alpha \quad (x \in E^n,\, n \in \mathbb{N})\,.$$

It is immediately checked that these formulas do indeed define type-p multi-norms. We shall identify $\|\cdot\|_n^{\min\text{-}p}$ and $\|\cdot\|_n^{\max\text{-}p}$ with some well known objects. First we list some properties of the weak-p summing norms that we shall need. By the results in [22, §2] we have the following.

Proposition 4.5.1. *Let E be a Banach space, let $1 \leq p < \infty$, and let $x = (x_1, \dots, x_n) \in E^n$.*

(i) *In the calculation of $\mu_{p,n}(x)$ we can replace $E'_{[1]}$ by any norming set in E'. In particular $\mu_{p,n}(x)$ calculated in E^n is the same as $\mu_{p,n}(x)$ calculated in $(E'')^n$.*

(ii) *We can avoid reference to the dual space altogether with the formula*

$$\mu_{p,n}(x) = \sup \left\{ \left\| \sum_{i=1}^{n} \alpha_i x_i \right\| : \|(\alpha_1, \dots, \alpha_n)\|_q \leq 1 \right\}, \tag{4.7}$$

where scalars $\alpha_1, \dots, \alpha_n$ lie in either $\mathbb{C}$ if E is a complex space, or $\mathbb{R}$ if E is a real space.

(iii) *For each Banach space F, and $T \in \mathcal{B}(E, F)$, we have*

$$\mu_{p,n}(Tx_1, \dots, Tx_n) \leq \|T\| \, \mu_{p,n}(x_1, \dots, x_n) . \qquad \square$$

Proposition 4.5.2 (min-max duality I)**.** *Let $1 \leq p \leq \infty$, let E be a type-p multi-normed space equipped with the maximal structure, and let F be a Banach space. Then the type-q multi-normed space $\mathcal{B}(E, F)^\times$ carries the minimal structure.*

Proof. Set $J = \mathcal{B}(E, F)$. Regard J as a type-q multi-normed space equipped with the minimal structure. By Proposition 4.4.9, $\mathcal{B}(J, F)^\times$ is a type-p multi-normed space. There is an embedding $E \to \mathcal{B}(J, F)^\times$, $x \mapsto \widehat{x}$ given by

$$\widehat{x}(T) = T(x) \quad (T \in J) .$$

This induces a type-p multi-norm structure on E, which must satisfy

$$\|\widehat{x}\|_n^\times \leq \|x\|_n^{\max\text{-}p} \quad (x \in E^n) .$$

Equivalently, for each $x = (x_1, \dots, x_n) \in E^n$ and $T = (T_1, \dots, T_n) \in \mathcal{B}(E, F)^n$, we have

$$\left\| \sum_{i=1}^{n} T_i(x_i) \right\|_F \leq \|x\|_n^{\max\text{-}p} \, \|T\|_n^{\min\text{-}q} .$$

Hence, for $T = (T_1, \dots, T_n) \in \mathcal{B}(E, F)^n$, we have

$$\|T\|_n^\times = \left\| \bigoplus T_i \right\|_{\mathcal{B}(E^n, F)} = \sup \left\{ \left\| \sum_{i=1}^{n} T_i(x_i) \right\|_F : \|x\|_n^{\max\text{-}p} \leq 1 \right\} \leq \|T\|_n^{\min\text{-}q} .$$

Therefore $\|T\|_n^\times = \|T\|_n^{\min\text{-}q}$, as required. $\square$

Proposition 4.5.3. *Let E be a Banach space, and let $p \in [1, \infty)$. Then the minimum type-p multi-norm over E is given by $\|x\|_n^{\min\text{-}p} = \mu_{p,n}(x)$ $(x \in E^n,\ n \in \mathbb{N})$.*

Proof. We first have to show that $(\mu_{p,n} : n \in \mathbb{N})$ is a type-p multi-norm over E. Take $m, n \in \mathbb{N}$, $a = (a_{ij}) \in \mathbb{M}_{m,n}$, $x = (x_1, \dots, x_n) \in E^n$, and $\lambda \in E'_{[1]}$. We set

$$y_i = \sum_{j=1}^{n} a_{ij} x_j \quad (i \in \mathbb{N}_m)$$

so that

$$a \cdot x = (y_1, \dots, y_m) \in E^m\,,$$

and we set

$$\alpha = (\langle x_1, \lambda \rangle, \dots, \langle x_n, \lambda \rangle) \in \mathbb{C}^n\,.$$

Then we have

$$\begin{aligned}\left(\sum_{i=1}^{m} |\langle y_i, \lambda \rangle|^p\right)^{1/p} &= \left(\sum_{i=1}^{m} \left|\sum_{j=1}^{n} a_{ij} \langle x_j, \lambda \rangle\right|^p\right)^{1/p} = \|a \cdot \alpha\|_{\ell^p_m} \\ &\leq \|a\|_p \|\alpha\|_{\ell^p_n} = \|a\|_p \left(\sum_{i=1}^{n} |\langle x_i, \lambda \rangle|^p\right)^{1/p}.\end{aligned}$$

Hence $\mu_{p,n}(a \cdot x) \leq \|a\|_p \, \mu_{p,n}(x)$ and $(\mu_{p,n} : n \in \mathbb{N})$ is a type-p multi-norm over E.

Now take $\alpha = (\alpha_i) \in \mathbb{C}^n$ with $\|\alpha\|_q \leq 1$. By Lemma 4.3.4 we have

$$\left\|\sum_{i=1}^{n} \alpha_i x_i\right\| \leq \|(x_1, \dots, x_n)\|_n^{\min\text{-}p}\,.$$

It follows from Proposition 4.5.1(ii) that $\mu_{p,n}(x_1, \dots, x_n) \leq \|(x_1, \dots, x_n)\|_n^{\min\text{-}p}$. Hence $\mu_{p,n} = \|\cdot\|_n^{\min\text{-}p}$ as required. □

Corollary 4.5.4. *Let E be a Banach space, and let $p \in (1, \infty]$. The maximum type-p multi-norm over E is given by*

$$\|x\|_n^{\max\text{-}p} = \sup\left\{\left|\sum_{i=1}^{n} \langle x_i, \lambda_i \rangle\right| : \lambda = (\lambda_1, \dots, \lambda_n) \in (E')^n,\ \mu_{q,n}(\lambda) \leq 1\right\},$$

where $x = (x_1, \dots, x_n) \in E^n$ and $n \in \mathbb{N}$.

Proof. This follows from Propositions 4.5.2 and 4.5.3 together with the Hahn–Banach theorem. □

Remark 4.5.5. The minimum type-∞ multi-norm and the maximum type-1 multi-norm over E are less interesting objects. For each $n \in \mathbb{N}$ and $x = (x_1, \dots, x_n) \in E^n$ we have

$$\|x\|_n^{\min\text{-}\infty} = \max_{i \in \mathbb{N}_n} \|x_i\| \quad \text{and} \quad \|x\|_n^{\max\text{-}1} = \sum_{i=1}^{n} \|x_i\|\,.$$

Corollary 4.5.6. *Let E be a Banach space, and let $p \in [1, \infty]$. Then for $x = (x_1, \dots, x_n) \in E^n$ we have*

$$\max_{i \in \mathbb{N}_n} \|x_i\| \le \|x\|_n^{\min\text{-}p} \le \left(\sum_{i=1}^n \|x_i\| \right)^{1/p} \le \|x\|_n^{\max\text{-}p} \le \sum_{i=1}^n \|x_i\| \,.$$

Proof. This follows from the characterizations in Proposition 4.5.3 and Corollary 4.5.4. □

We have proved easily in Proposition 4.5.2 that the dual of the maximum type-p multi-norm over E is the minimum type-q multi-norm over E'. The fact that the dual of the minimum type-p multi-norm over E is the maximum type-q multi-norm over E' seems to lie a little deeper and requires the Principle of Local Reflexivity.

Lemma 4.5.7. *Let E be a Banach space, let $n \in \mathbb{N}$, and let μ be a norm on $(E'')^n$ such that*

$$\mu(T(\Phi_1), \dots, T(\Phi_n)) \le \|T\| \, \mu(\Phi_1, \dots, \Phi_n) \quad (T \in \mathcal{B}(E'', E)) \,.$$

Then for each $\lambda \in (E')^n$ we have

$$\sup \, \{|\langle \lambda, \Phi \rangle| : \Phi \in (E'')^n, \, \mu(\Phi) \le 1\} = \sup \, \{|\langle x, \lambda \rangle| : x \in E^n, \, \mu(x) \le 1\} \,.$$

Proof. We denote the left and right hand sides of this equation by $\|\lambda\|$ and $|||\lambda|||$ respectively. Clearly $\|\lambda\| \ge |||\lambda|||$. Take $\varepsilon > 0$ and $\Phi = (\Phi_1, \dots, \Phi_n) \in (E'')^n$ with $\mu(\Phi) \le 1$ and

$$\|\lambda\| - \varepsilon \le \left| \sum_{i=1}^n \langle \lambda_i, \Phi_i \rangle \right| \,.$$

By the Principle of Local Reflexivity there exists $T \in \mathcal{B}(E'', E)$ with $\|T\| \le 1 + \varepsilon$ and

$$\langle T(\Phi_i), \lambda_i \rangle = \langle \lambda_i, \Phi_i \rangle \quad (i \in \mathbb{N}_n) \,.$$

Set $x = (T(\Phi_1), \dots, T(\Phi_n)) \in E^n$. By hypothesis

$$\mu(x) \le (1 + \varepsilon) \mu(\Phi) \le 1 + \varepsilon \,.$$

Now we have

$$|||\lambda||| \ge \frac{1}{1+\varepsilon} \left| \sum_{i=1}^n \langle T(\Phi_i), \lambda_i \rangle \right| = \frac{1}{1+\varepsilon} \left| \sum_{i=1}^n \langle \lambda_i, \Phi_i \rangle \right| \ge \frac{1}{1+\varepsilon} \left(\|\lambda\| - \varepsilon \right) \,.$$

This is true for each $\varepsilon > 0$. Therefore $|||\lambda||| \ge \|\lambda\|$ as required. □

Proposition 4.5.8 (min-max duality II)**.** *Let $1 \le p \le \infty$, and let E be a type-p multi-normed space equipped with the minimal structure. Then the type-q multi-normed space $E^\times$ carries the maximal structure.*

Proof. Take $\lambda \in (E')^n$. By Corollary 4.5.4 we have

$$\|\lambda\|_n^{\max\text{-}p} = \sup \{|\langle \lambda, \Phi \rangle| : \Phi \in (E'')^n,\ \mu_{q,n}(\Phi) \le 1\}\ .$$

By definition we have

$$\|\lambda\|_n^{\times} = \sup \{|\langle x, \lambda \rangle| : x \in E^n,\ \mu_{q,n}(x) \le 1\}\ .$$

By Lemma 4.5.7 these two quantities are equal. □

Proposition 4.5.9. *Let E be a Banach space, and let $1 \le p \le \infty$. Then, for each $n \in \mathbb{N}$, there are isometric isomorphisms of Banach spaces:*

(i) $(E^n, \|\cdot\|_n^{\max\text{-}p}) \cong (\ell_n^p \otimes E, \|\cdot\|_\pi)$; *and*

(ii) $(E^n, \|\cdot\|_n^{\min\text{-}p}) \cong \mathcal{B}(\ell_n^q, E)$.

Proof. (i) We define a linear operator $\theta : \ell_n^p \otimes E \to E^n$ by

$$\theta\left((\alpha_i) \otimes x\right) = (\alpha_1 x, \dots, \alpha_n x) \quad \left((\alpha_i) \in \ell_n^p,\ x \in E\right).$$

Take $z = \sum_{i=1}^N y_i \otimes x_i \in \ell_n^p \otimes E$, where $y_i = (\alpha_{k,i})_{k=1}^n \in \ell_n^p$ $(i \in \mathbb{N}_N)$. Then by Lemma 4.3.2, we have

$$\|\theta(z)\|_n^{\max\text{-}p} \le \sum_{i=1}^N \|(\alpha_{1,i}x_i, \dots, \alpha_{n,i}x_i)\|_n^{\max\text{-}p} \le \sum_{i=1}^N \|y_i\|_{\ell^p} \|x_i\|\ .$$

Hence $\|\theta\| \le 1$.

Now we define a linear operator $\theta^{-1} : E^n \to \ell_n^p \otimes E$ by

$$\theta^{-1}(x_1, \dots, x_n) = \sum_{i=1}^n \delta_i \otimes x_i \quad \left((x_1, \dots, x_n) \in E^n\right).$$

Then θ^{-1} is a two-sided inverse to θ. For each $x = (x_1, \dots, x_n) \in E^n$ and each $T \in \mathcal{B}(\ell_n^p, E') = (\ell_n^p \otimes E)'$ we have

$$\begin{aligned} \left|\left\langle \theta^{-1}(x), T \right\rangle\right| &= \left| \left\langle \sum_{i=1}^n \delta_i \otimes x_i, T \right\rangle \right| = \left| \sum_{i=1}^n \langle x_i, T(\delta_i) \rangle \right| \\ &\le \|x\|_n^{\max\text{-}p} \mu_{q,n}(T(\delta_1), \dots, T(\delta_n)) \\ &= \|x\|_n^{\max\text{-}p} \|T\|\ . \end{aligned}$$

Hence

$$\left\|\theta^{-1}(x)\right\| = \sup \left\{ \left|\left\langle \theta^{-1}(x), T \right\rangle\right| : \|T\| \le 1 \right\} \le \|x\|_n^{\max\text{-}p}\ .$$

Therefore the two spaces are isometrically isomorphic.

(ii) This is [9, Proposition 2.2]. □

Proposition 4.5.10. *Let $1 \leq p \leq \infty$, let E be a Banach space, and let F be a type-p multi-normed space equipped with the minimal structure. Then the type-p multi-normed space $\mathcal{B}(E,F)^\dagger$ carries the minimal structure.*

Proof. Take $T_1, \ldots, T_n \in \mathcal{B}(E,F)$. Then by Proposition 4.5.3 and Proposition 4.5.1(ii) we have

$$\begin{aligned}
\|(T_1,\ldots,T_n)\|_n^{\min\text{-}p} &= \sup_{\alpha\in(\ell^q)_{[1]}} \left\{ \left\| \sum_{i=1}^n \alpha_i T_i \right\|_{\mathcal{B}(E,F)} \right\} \\
&= \sup_{\alpha\in(\ell^q)_{[1]}} \sup_{x\in E_{[1]}} \left\{ \left\| \sum_{i=1}^n \alpha_i T_i(x) \right\|_F \right\} \\
&= \sup_{x\in E_{[1]}} \sup_{\alpha\in(\ell^q)_{[1]}} \left\{ \left\| \sum_{i=1}^n \alpha_i T_i(x) \right\|_F \right\} \\
&= \sup_{x\in E_{[1]}} \|(T_1(x),\ldots,T_n(x))\|_{F^n} \\
&= \left\| \prod T_i \right\|_{\mathcal{B}(E,F^n)} = \|(T_1,\ldots,T_n)\|_n^\dagger .
\end{aligned}$$

The result follows. □

4.6 Type-∞ and type-1 multi-normed spaces

Our main interest is in type-∞ and type-1 multi-normed spaces. From now on we shall follow the terminology of [7] and refer to a type-∞ multi-norm simply as a *multi-norm*, and a type-1 multi-norm as a *dual multi-norm*. The terms *multi-normed space* and *dual multi-normed space* will refer to type-∞ and type-1 multi-normed spaces respectively. The term *maximum multi-norm* will refer to the maximum type-1 multi-norm etc,.

4.6.1 A characterization of dual multi-normed spaces

Multi-normed spaces have a simple axiomatic characterization. Indeed, this is how multi-normed spaces are defined in [7].

Proposition 4.6.1 ([7, Theorem 2.34])**.** *Let $(E, \|\cdot\|)$ be a Banach space, and let $(\|\cdot\|_n : n \in \mathbb{N})$ be a sequence such that $\|\cdot\|_n$ is a norm on E^n for each $n \in \mathbb{N}$, with $\|\cdot\|_1 = \|\cdot\|$ on $E = E^1$. Then the sequence $(\|\cdot\|_n : n \in \mathbb{N})$ is a multi-norm if and only if axioms* (A1)-(A3) *hold, and for each $n \geq 2$ and $x_1, \ldots, x_{n-1} \in E$ the following additional axiom holds:*

(A4) $\|(x_1,\ldots,x_{n-2},x_{n-1},x_{n-1})\|_n = \|(x_1,\ldots,x_{n-2},x_{n-1})\|_{n-1}$. □

Here we show that dual multi-normed spaces admit a similar characterization. Let E be a Banach space, and let $(\|\cdot\|_n : n \in \mathbb{N})$ be a sequence of norms on the spaces $(E^n : n \in \mathbb{N})$. Consider the following axiom, required to hold for all $n \geq 2$ and $x_1, \ldots, x_{n-1} \in E$:

(B4) $\|(x_1, \ldots, x_{n-2}, x_{n-1}, x_{n-1})\|_n = \|(x_1, \ldots, x_{n-2}, 2x_{n-1})\|_{n-1}$.

Lemma 4.6.2. *Let E be a special-normed space which satisfies axiom* (B4). *Let $n \geq 2$, and let $x_1, \ldots, x_n \in E$. Then*

$$\|(x_1, \ldots, x_{n-2}, x_{n-1} + x_n)\|_{n-1} \leq \|(x_1, \ldots, x_{n-2}, x_{n-1}, x_n)\|_n \,.$$

Proof. We have

$$\begin{aligned}
&\|(x_1, \ldots, x_{n-2}, x_{n-1} + x_n)\|_{n-1} \\
&\quad = \|(x_1, \ldots, x_{n-2}, (x_{n-1} + x_n)/2, (x_{n-1} + x_n)/2)\|_n \quad \text{by (B4)} \\
&\quad = 1/2\,\|(x_1, \ldots, x_{n-2}, x_{n-1}, x_n) + (x_1, \ldots, x_{n-2}, x_n, x_{n-1})\|_n \\
&\quad \leq \|(x_1, \ldots, x_{n-2}, x_{n-1}, x_n)\|_n \quad \text{by (A1)}\,.
\end{aligned}$$

□

Lemma 4.6.3. *Let E be a special-normed space which satisfies axiom* (B4). *Let $m, n \in \mathbb{N}$, let $X_1, \ldots, X_m$ be pairwise disjoint subsets of $\mathbb{N}_n$, and let $x_1, \ldots, x_n \in E$. Then*

$$\left\| \left(\sum_{i \in X_1} x_i, \ldots, \sum_{i \in X_m} x_i \right) \right\|_m \leq \|(x_1, \ldots, x_n)\|_n \,.$$

Proof. We interpret the sum over the empty set as 0. If none of the sets $X_1, \ldots, X_m$ are empty then the result follows by repeated use of Lemma 4.6.2. If some of the sets are empty (this must be the case if $m > n$) then we have to use Lemma 4.6.2 and Lemma 4.2.6. □

Theorem 4.6.4. *Let $(E, \|\cdot\|)$ be a Banach space, and let $(\|\cdot\|_n : n \in \mathbb{N})$ be a sequence such that $\|\cdot\|_n$ is a norm on E^n for each $n \in \mathbb{N}$, with $\|\cdot\|_1 = \|\cdot\|$ on $E = E^1$. Then the following are equivalent:*

(i) *Axioms* (A1)-(A3) *and* (B4) *hold.*

(ii) *$\|a \cdot x\|_m \leq \|a\|_1 \|x\|_n$ for each column-special matrix $a \in \mathbb{M}_{m,n}$, each $x \in E^n$, and each $m, n \in \mathbb{N}$.*

(iii) *$(\|\cdot\|_n : n \in \mathbb{N})$ is a dual multi-norm.*

Proof. (i) ⇒ (ii) Let $a = (a_{ij}) \in \mathbb{M}_{m,n}$ be a column-special matrix, so that a has at most one non-zero entry in each column. Let the non-zero entry in column j (if it exists) be in row $i(j)$. This entry is $a_{i(j),j}$. If there is no non-zero entry in column j then we set $a_{i(j),j} = 0$. Set

$$X_i = \{j \in \mathbb{N}_n : i(j) = i, a_{i(j),j} \neq 0\} \quad (i \in \mathbb{N}_m)\,,$$

so that X_i is the set of columns of a that have a non-zero entry in row i. The sets (X_i) are pairwise disjoint.

Take $x = (x_1, \ldots, x_n) \in E^n$. Then we have

$$\begin{aligned}\|a \cdot x\|_m &= \left\| \left(\sum_{i \in X_1} a_{1,i} x_i, \ldots, \sum_{i \in X_m} a_{m,i} x_i \right) \right\|_m \\ &\leq \left\| (a_{i(1),1} x_1, \ldots, a_{i(n),n} x_n) \right\|_n \quad \text{by Lemma 4.6.3} \\ &\leq \max_{j \in \mathbb{N}_n} \left| a_{i(j),j} \right| \|(x_1, \ldots, x_n)\|_n \quad \text{by (A2)} \\ &= \|a\|_1 \|x\|_n \quad \text{by (4.1)}\,.\end{aligned}$$

Therefore (ii) holds.

(ii) $\Rightarrow$ (iii) Take $a \in \mathbb{M}_{m,n}$. We can write a as a sum of column-special matrices $a = \sum_{r=1}^k a_r$ with $\|a\|_1 = \sum_{r=1}^k \|a_r\|_1$ (see [7, §1.6]). For each $x \in E^n$ we have

$$\|a \cdot x\|_m \leq \sum_{r=1}^k \|a_r \cdot x\|_m \leq \sum_{r=1}^k \|a_r\|_1 \|x\|_n = \|a\|_1 \|x\|_n\,,$$

as required.

(iii) $\Rightarrow$ (i) By Proposition 4.2.2 (A1)-(A3) hold, and by Lemma 4.3.3 (B4) holds. □

4.6.2 The standard p-multi-norm

In this section we introduce a general class of multi-norms, which will play an important role in the next chapter.

Proposition 4.6.5. *Let E be a Banach space, and let $1 \leq p < \infty$. For each $n \in \mathbb{N}$ we define a norm on E^n by*

$$\|x\|_n = \sup \left\{ \left(\sum_{i=1}^n |\langle x_i, \lambda_i \rangle|^p \right)^{1/p} : \lambda \in (E')^n,\, \mu_{1,n}(\lambda) \leq 1 \right\},$$

where $x = (x_1, \ldots, x_n) \in E^n$. Then the family $(\|\cdot\|_n : n \in \mathbb{N})$ is a multi-norm.

Proof. It is clear that the family $(\|\cdot\|_n : n \in \mathbb{N})$ is a special-norm. We shall verify axiom (A4).

Let q be the conjugate index to p. Let $n \geq 2$, let $x_1, \ldots, x_{n-1} \in E$, and let $\varepsilon > 0$. Set $x = (x_1, \ldots, x_{n-2}, x_{n-1}, x_{n-1}) \in E^n$. There exists $\lambda_1, \ldots, \lambda_n \in E'$ with $\mu_{n,1}(\lambda_1, \ldots, \lambda_n) \leq 1$, and such that

$$\|x\|_n - \varepsilon < \left(\sum_{i=1}^{n-2} |\langle x_i, \lambda_i \rangle|^p + |\langle x_{n-1}, \lambda_{n-1} \rangle|^p + |\langle x_{n-1}, \lambda_n \rangle|^p \right)^{1/p}.$$

By the Hahn–Banach theorem there exists $\alpha, \beta \in \mathbb{C}$ with $\gamma = (|\alpha|^q + |\beta|^q)^{1/q} \leq 1$ and

$$|\langle x_{n-1}, \lambda_{n-1}\rangle|^p + |\langle x_{n-1}, \lambda_{n-1}\rangle|^p = \langle x_{n-1}, \alpha\lambda_{n-1} + \beta\lambda_n\rangle^p \ .$$

Since $\{\mu_{p,n} : n \in \mathbb{N}\}$ is a type-p multi-norm, by Lemma 4.3.4, we have

$$\begin{aligned}\mu_{p,n-1}(\lambda_1, \ldots, \lambda_{n-2}, \alpha\lambda_{n-1} + \beta\lambda_n) &\leq \mu_{p,n}(\lambda_1, \ldots, \lambda_{n-2}, \gamma\lambda_{n-1}, \gamma\lambda_n)\\ &\leq \max\{1, \gamma\}\mu_{p,n}(\lambda_1, \ldots, \lambda_n) \leq 1 \, .\end{aligned}$$

Hence

$$\|x\|_n - \varepsilon < \left(\sum_{i=1}^{n-2} |\langle x_i, \lambda_i\rangle|^p + \langle x_{n-1}, \alpha\lambda_{n-1} + \beta\lambda_n\rangle^p\right)^{1/p} \leq \|(x_1, \ldots, x_{n-1})\|_{n-1} \ .$$

The reverse inequality is Lemma 4.2.8(ii). □

Definition 4.6.6. Let E be a Banach space, and let $1 \leq p < \infty$. Then the *standard p-multi-norm* over E is the multi-norm $(\|\cdot\|_n : n \in \mathbb{N})$ described above.

Remark 4.6.7. The standard 1-multi-norm is just the maximum multi-norm.

Lemma 4.6.8. *Let E be a Banach space, and let $1 \leq p_1, p_2 < \infty$. Then for each $n \in \mathbb{N}$ and $\lambda \in (E')^n$ we have*

$$\begin{aligned}&\sup\left\{\left(\sum_{n=1}^{n} |\langle \lambda_i, \Phi_i\rangle|^{p_2}\right)^{1/p_2} : \Phi \in (E'')^n,\ \mu_{p_1,n}(\Phi) \leq 1\right\}\\ &\qquad = \sup\left\{\left(\sum_{n=1}^{n} |\langle x_i, \lambda_i\rangle|^{p_2}\right)^{1/p_2} : x \in E^n,\ \mu_{p_1,n}(x) \leq 1\right\} .\end{aligned}$$

Proof. This follows from the Principle of Local Reflexivity in exactly the same way as the proof of Lemma 4.5.7. □

4.6.3 The standard p-multi-norm over $L^1(\Omega)$

In this section we give a concrete description of the standard p-multi-norm over the Banach spaces $L^1(\Omega)$ and $M(\Omega)$. We shall identify this norm with the *standard $(1,p)$-multi-norm* defined in [7, Definition 4.7]. Hence we shall denote the standard p-multi-norm on these spaces by $(\|\cdot\|_n^{(1,p)} : n \in \mathbb{N})$. The result is based on the following identification of $\mu_{1,n}$.

Proposition 4.6.9 ([22, 2.6])**.** *Let K be a compact space, and let $\lambda_1, \ldots, \lambda_n \in C(K)$. Then*

$$\mu_{1,n}(\lambda_1, \ldots, \lambda_n) = \left|\sum_{i=1}^{n} |\lambda_i|\right|_K \ .$$

□

Let Ω be a locally compact space. Then $M(\Omega)'$ is isometrically isomorphic to $C(\widetilde{\Omega})$ for some compact space $\widetilde{\Omega}$ called the *hyper-stonean cover of* Ω, (this follows from general C*-algebra theory as in [39, III.2.3], a direct proof is given in [6]). There is an isometric embedding $\kappa : M(\Omega) \to M(\widetilde{\Omega})$ which identifies $M(\Omega)$ with the closed subspace of $M(\widetilde{\Omega})$ consisting of the *normal* measures on $\widetilde{\Omega}$. Thus we shall interpret $\mu \in M(\Omega)$ as a measure on $\widetilde{\Omega}$. The duality between $M(\Omega)$ and $C(\widetilde{\Omega})$ is then given by

$$\langle \mu, \lambda \rangle = \int_{\widetilde{\Omega}} \lambda \, \mathrm{d}\mu \quad (\mu \in M(\Omega),\ \lambda \in C(\widetilde{\Omega}))\,.$$

Let $\mu, \nu \in M(\Omega)$ be positive measures. Then we define $\mu \vee \nu \in M(\Omega)$ by

$$(\mu \vee \nu)(E) = \sup \{\mu(E_1) + \nu(E_2) : (E_1, E_2) \text{ a measurable partition of } E\}\,,$$

where $E \in \mathcal{B}_\Omega$.

Theorem 4.6.10. *Let Ω be a locally compact space. Then the maximum multi-norm over $M(\Omega)$ is given by*

$$\|(\mu_1, \ldots, \mu_n)\|_n^{\max} = \| |\mu_1| \vee \cdots \vee |\mu_n| \| \quad (\mu_1, \ldots, \mu_n \in M(\Omega))\,.$$

Proof. Take $n \in \mathbb{N}$ and $\mu_1, \ldots, \mu_n \in M(\Omega)$. For $\lambda_1, \ldots, \lambda_n \in C(\widetilde{\Omega})$ we have

$$\begin{aligned}
\left| \sum_{i=1}^n \langle \mu_i, \lambda_i \rangle \right| &\le \sum_{i=1}^n |\langle \mu_i, \lambda_i \rangle| \le \sum_{i=1}^n \langle |\mu_i|, |\lambda_i| \rangle \le \sum_{i=1}^n \langle |\mu_1| \vee \cdots \vee |\mu_n|, |\lambda_i| \rangle \\
&= \left\langle |\mu_1| \vee \cdots \vee |\mu_n|, \sum_{i=1}^n |\lambda_i| \right\rangle \le \| |\mu_1| \vee \cdots \vee |\mu_n| \| \left| \sum_{i=1}^n |\lambda_i| \right|_{\widetilde{\Omega}} \\
&= \| |\mu_1| \vee \cdots \vee |\mu_n| \| \, \mu_{1,n}(\lambda_1, \ldots, \lambda_n)\,.
\end{aligned}$$

Therefore, by Corollary 4.5.3 we have

$$\begin{aligned}
\|(\mu_1, \ldots, \mu_n)\|_n^{\max} &= \sup \left\{ \left| \sum_{i=1}^n \langle \mu_i, \lambda_i \rangle \right| : \mu_{1,n}(\lambda_1, \ldots, \lambda_n) \le 1 \right\} \\
&\le \| |\mu_1| \vee \cdots \vee |\mu_n| \|\,.
\end{aligned}$$

Hence $\|(\mu_1, \ldots, \mu_n)\|_n^{\max} = \| |\mu_1| \vee \cdots \vee |\mu_n| \|$, as required. □

Let Ω be a locally compact space, and let $X \subset \Omega$ be a measurable set. We define a projection $P_X : M(\Omega) \to M(\Omega)$ by

$$P_X(\mu)(E) = \mu(X \cap E) \quad (E \in \mathcal{B}_\Omega)\,.$$

When restricted to $L^1(\Omega)$ this map has the form

$$P_X(f) = \chi_X f \quad (f \in L^1(\Omega))\,.$$

Proposition 4.6.11. *Let Ω be a locally compact space, and let $\mu_1, \dots, \mu_n \in M(\Omega)$. Then:*

$$\| |\mu_1| \vee \cdots \vee |\mu_n| \| = \sup_{\mathbf{X}} \sum_{i=1}^{n} \| P_{X_i}(\mu_i) \| \ ,$$

where the supremum is taken over all measurable partitions $\mathbf{X} = (X_1, \dots, X_n)$ of Ω.

Proof. Take $\mu_1, \dots, \mu_n \in M(\Omega)$. For each measurable partition $\mathbf{X} = (X_1, \dots, X_n)$ of Ω, we have

$$\sum_{i=1}^{n} \| P_{X_i}(\mu_i) \| = \sum_{i=1}^{n} |\mu_i| (X_i) \leq \sum_{i=1}^{n} (|\mu_1| \vee \cdots \vee |\mu_n|)(X_i) = \| |\mu_1| \vee \cdots \vee |\mu_n| \| \ .$$

Hence $\sup_{\mathbf{X}} \sum_{i=1}^{n} \| P_{X_i}(\mu_i) \| \leq \| |\mu_1| \vee \cdots \vee |\mu_n| \|$.

We shall prove that for each $n \geq 2$ and $\mu_1, \dots, \mu_n \in M(\Omega)$, there exists a measurable partition $(X_1, \dots, X_n)$ of Ω with

$$\| |\mu_1| \vee \cdots \vee |\mu_n| \| = \sum_{i=1}^{n} \| P_{X_i}(\mu_i) \| \ .$$

This is most easily done by induction on n.

For positive measures $\mu, \nu \in M(\Omega)$, by the Hahn decomposition theorem ([3, Theorem 4.1.4]), there exists a set $(\mu \geq \nu) \in \mathcal{B}_\Omega$ with the property that $\mu(E) \geq \nu(E)$ for all measurable subsets $E \subset (\mu \geq \nu)$, and $\mu(E) \leq \nu(E)$ for all measurable subsets $E \subset \Omega \setminus (\mu \geq \nu)$.

Consider the case $n = 2$, and let $\mu_1, \mu_2 \in M(\Omega)$. Set $X_1 = (|\mu_1| \geq |\mu_2|)$ and $X_2 = \Omega \setminus X_1$. Then we have

$$\begin{aligned} \| |\mu_1| \vee |\mu_2| \| &= |\mu_1| \vee |\mu_2| (X_1) + |\mu_1| \vee |\mu_2| (X_2) \\ &= |\mu_1| (X_1) + |\mu_2| (X_2) \\ &= \| P_{X_1}(\mu_1) \| + \| P_{X_2}(\mu_2) \| \ . \end{aligned}$$

Now assume that the result holds for some $n \in \mathbb{N}$, and take $\mu_1, \dots, \mu_{n+1} \in M(\Omega)$. Set $\mu = |\mu_1| \vee \cdots \vee |\mu_n|$ and $X = (\mu \geq \mu_{n+1})$. By the inductive hypothesis there is a measurable partition $(Y_1, \dots, Y_n)$ of Ω with

$$\| |P_X(\mu_1)| \vee \cdots \vee |P_X(\mu_n)| \| = \sum_{i=1}^{n} \| P_{Y_i}(P_X(\mu_i)) \| = \sum_{i=1}^{n} \| P_{Y_i \cap X}(\mu_i) \| \ .$$

Then we have

$$\begin{aligned} \| |\mu_1| \vee \cdots \vee |\mu_{n+1}| \| &= \| \mu \vee |\mu_{n+1}| \| = \| P_X(\mu) \| + \left\| P_{\Omega \setminus X}(\mu_{n+1}) \right\| \\ &= \| |P_X(\mu_1)| \vee \cdots \vee |P_X(\mu_n)| \| + \left\| P_{\Omega \setminus X}(\mu_{n+1}) \right\| \\ &= \sum_{i=1}^{n} \| P_{Y_i \cap X}(\mu_i) \| + \left\| P_{\Omega \setminus X}(\mu_{n+1}) \right\| \ , \end{aligned}$$

where the sets $(Y_1 \cap X, \dots, Y_n \cap X, \Omega \setminus X)$ form a measurable partition of Ω. By induction, the result follows. ∎

Let K be a compact space, and take $n \in \mathbb{N}$. We define D_n to be the set of $(\lambda_1, \dots, \lambda_n) \in C(K)^n$ such that $|\lambda_i|_K \leq 1$ $(i \in \mathbb{N}_n)$, and the sets supp $\lambda_1, \dots,$ supp λ_n are pairwise disjoint.

Corollary 4.6.12. *Let K be a compact space. Then $(C(K)^n, \mu_{1,n})_{[1]} = \overline{\langle D_n \rangle}$, where the closure is in the weak-$*$ topology.*

Proof. Set $B_n = (C(K)^n, \mu_{1,n})_{[1]}$. It is easily seen using Proposition 4.6.9 that $\overline{\langle D_n \rangle} \subset B_n$. Assume towards a contradiction that there exists $\varphi \in B_n \setminus \overline{\langle D_n \rangle}$. Since $\overline{\langle D_n \rangle}$ is a balanced set, by Theorem 1.1.1 there exists $\mu = (\mu_1, \dots, \mu_n) \in M(K)^n$ with

$$\langle \lambda, \mu \rangle \leq 1 \quad (\lambda \in \overline{\langle D_n \rangle}) \quad \text{and} \quad \langle \varphi, \mu \rangle > 1 \,.$$

By Proposition 4.6.11, we have

$$\| |\mu_1| \vee \cdots \vee |\mu_n| \| = \sup_{\mathbf{X}} \sum_{i=1}^{n} \| P_{X_i}(\mu_i) \| = \sup_{\lambda \in D_n} \left| \sum_{i=1}^{n} \langle \lambda_i, \mu_i \rangle \right| \leq 1 < \|\mu\|_n^{\max} \,,$$

which is a contradiction of Theorem 4.6.10. Therefore $B_n = \overline{\langle D_n \rangle}$. □

Theorem 4.6.13. *Let Ω be a locally compact space, and let $1 \leq p < \infty$. Then the standard p-multi-norm over $M(\Omega)$ is given by*

$$\|(\mu_1, \dots, \mu_n)\|_n^{(1,p)} = \sup_{\mathbf{X}} \left(\sum_{i=1}^{n} \| P_{X_i}(\mu_i) \|^p \right)^{1/p} \qquad (\mu_1, \dots, \mu_n \in M(\Omega)) \,.$$

where the supremum is taken over all measurable partitions $\mathbf{X} = (X_1, \dots, X_n)$ of Ω.

Proof. Take $\mu_1, \dots, \mu_n \in M(\Omega)$. By Corollary 4.6.12 we have

$$\sup \left\{ \left(\sum_{i=1}^{n} |\langle \mu_i, \lambda_i \rangle|^p \right)^{1/p} : \lambda \in (C(\widetilde{\Omega})^n, \mu_{1,n})_{[1]} \right\} = \sup \left\{ \left(\sum_{i=1}^{n} |\langle \mu_i, \lambda_i \rangle|^p \right)^{1/p} : \lambda \in D_n \right\} .$$

This is a reformulation of the stated equality. □

The following remark is contained in [7, Example 4.9]. Let Ω be a locally compact space. Then $L^1(\Omega)''$ is isometrically isomorphic to $M(K)$ for a certain compact space K. Let $X \in \mathcal{B}_\Omega$, and let $P_X \in \mathcal{B}(L^1(\Omega))$ be the projection onto $L^1(X)$. Then $P_X'' \in \mathcal{B}(M(K))$ can be identified with $P_{\widetilde{X}} \in \mathcal{B}(M(K))$ for some measurable set $\widetilde{X} \subset K$. The collection $\{\widetilde{X} : X \in \mathcal{B}_\Omega\}$ forms a base of clopen sets for the topology on K. Hence by Theorem 4.6.13 we have the following.

Proposition 4.6.14. *Let Ω be a measure space, and take $1 \leq p < \infty$. Then the standard p-multi-norm over $L^1(\Omega)''$ is given by*

$$\|(\Phi_1, \ldots, \Phi_n)\|_n^{(1,p)} = \sup_{\mathbf{X}} \left(\sum_{i=1}^{n} \left\| P''_{X_i}(\Phi_i) \right\|^p \right)^{1/p} \quad (\Phi_1, \ldots, \Phi_n \in L^1(\Omega)''),$$

where the supremum is taken over all measurable partitions $\mathbf{X} = (X_1, \ldots, X_n)$ of Ω. □

4.6.4 Extensions of multi-norms

Let F be a Banach space, and let E be a multi-normed space. For each $n \in \mathbb{N}$ we define a norm $\|\cdot\|_n^{\mathcal{B}}$ on the space F^n by setting

$$\|(x_1, \ldots, x_n)\|_n^{\mathcal{B}} = \sup_{U} \|(U(x_1), \ldots, U(x_n))\|_n \quad (x_1, \ldots, x_n \in F),$$

where the supremum is taken over all $U \in \mathcal{B}(F, E)_{[1]}$. It is immediately checked that this defines a multi-norm over F, and that

$$\mathcal{M}(F, E) = \mathcal{B}(F, E) \quad \text{with} \quad \|T\|_{mb} = \|T\| \quad (T \in \mathcal{M}(F, E)). \tag{4.8}$$

Let $(|||\cdot|||_n : n \in \mathbb{N})$ be a multi-norm over F such that (4.8) holds. Then it is clear that $\|\cdot\|_n^{\mathcal{B}} \leq |||\cdot|||_n \ (n \in \mathbb{N})$.

Proposition 4.6.15. *Let F be a Banach space, and let E be a multi-normed space.*

(ii) *In the definition of $\|\cdot\|_n^{\mathcal{B}}$ we can replace $\mathcal{B}(F, E)_{[1]}$ by any* wo*-dense subset of $\mathcal{B}(F, E)_{[1]}$.*

(ii) *Suppose that either E or F has the metric approximation property. Then in the definition of $\|\cdot\|_n^{\mathcal{B}}$ we can replace $\mathcal{B}(F, E)_{[1]}$ by $\mathcal{F}(F, E)_{[1]}$.*

Proof. (i) Let $X \subset \mathcal{B}(F, E)_{[1]}$ be a wo-dense set in $\mathcal{B}(F, E)_{[1]}$. Take $n \in \mathbb{N}$ and $x = (x_1, \ldots, x_n) \in F^n$. We set $\|x\|_n^X = \sup_{U \in X} \|(U(x_1), \ldots, U(x_n))\|_n$. Fix $\varepsilon > 0$. By th Hahn–Banach theorem, there exists $U \in \mathcal{B}(F, E)_{[1]}$ and $\lambda = (\lambda_1, \ldots, \lambda_n) \in (E')^n$ with $\|\lambda\|_n^{\times} \leq 1$, such that

$$\|x\|_n^{\mathcal{B}} \leq \left| \sum_{i=1}^{n} \langle U(x_i), \lambda_i \rangle \right| + \varepsilon/2 .$$

Take $T \in X$ with $|\langle U(x_i) - T(x_i), \lambda_i \rangle| < \varepsilon/2n \ (i \in \mathbb{N}_n)$. Then we have

$$\begin{aligned} \|x\|_n^{\mathcal{B}} &\leq \left| \sum_{i=1}^{n} \langle T(x_i), \lambda_i \rangle \right| + \left| \sum_{i=1}^{n} \langle U(x_i) - T(x_i), \lambda_i \rangle \right| + \varepsilon/2 \\ &\leq \|x\|_n^X + \sum_{i=1}^{n} |\langle U(x_i) - T(x_i), \lambda_i \rangle| + \varepsilon/2 \leq \|x\|_n^X + \varepsilon . \end{aligned}$$

The result follows.

(ii) This follows from part (i) since in either case $\mathcal{F}(F,E)_{[1]}$ is wo-dense (in fact, so-dense) in $\mathcal{B}(F,E)_{[1]}$. □

Definition 4.6.16. Let F be a Banach space, let E be a multi-normed space. Then the multi-norm $(\|\cdot\|_n^{\mathcal{B}} : n \in \mathbb{N})$ described above is the *extension* to F of the multi-norm on E.

Example 4.6.17 ([7, Example 4.2])**.** Let Ω be a measure space, and let $1 \le p < \infty$. For $n \in \mathbb{N}$ and $f_1, \dots, f_n \in L^p(\Omega)$ we define

$$\|(f_1,\dots,f_n)\|_n^{\mathrm{lat}} = \sup_{\mathbf{X}} \left(\sum_{i=1}^n \|\chi_{X_i} f_i\|_p^p \right)^{1/p},$$

where the supremum is taken over all measurable partitions $\mathbf{X} = (X_1, \dots, X_n)$ of Ω. The family $(\|\cdot\|_n^{\mathrm{lat}} : n \in \mathbb{N})$ is a multi-norm over $L^p(\Omega)$ called the *lattice multi-norm*. We have

$$\|(f_1,\dots,f_n)\|_n^{\mathrm{lat}} = \||f_1| \vee \cdots \vee |f_n|\|_p .$$

The lattice multi-norm can be defined on any Banach lattice, see [7] for more about this structure.

4.6.5 The extension of the lattice multi-norm on $L^p(\Omega)$

Let F be a Banach space, and let $1 \le p < \infty$ with conjugate index q. Let $(\Omega, \mathcal{F}, m)$ be a measure space. We shall identify the extension to F of the lattice multi-norm on $L^p(\Omega)$. From now on the notation $(\|\cdot\|_n^{\mathcal{B}} : n \in \mathbb{N})$ is reserved for this extension. We suppose that the measure space $(\Omega, \mathcal{F}, m)$ satisfies the following condition: there exists an infinite collection of pairwise disjoint subsets $\{X_n : n \in \mathbb{N}\} \subset \mathcal{F}$ with $0 < m(X_n) < \infty \ (n \in \mathbb{N})$. This condition ensures that the space $L^p(\Omega)$ is infinite dimensional.

Let D_n denote the collection of all $(f_1,\dots,f_n) \in (L^q(\Omega)_{[1]})^n$ such that the sets $\operatorname{supp} f_1, \dots, \operatorname{supp} f_n$ are pairwise disjoint. For a Banach space X, set

$$B_n(X) = \left\{ (U(f_1),\dots,U(f_n)) : U \in \mathcal{B}(L^q(\Omega), X)_{[1]},\ (f_1,\dots,f_n) \in D_n \right\} \subset X^n .$$

Lemma 4.6.18. *We have $B_n(X) = \{x \in X^n : \mu_{p,n}(x) \le 1\}$.*

Proof. Set $C_n(X) = \{x \in X^n : \mu_{p,n}(\lambda) \le 1\}$.

Take $U \in \mathcal{B}(L^q(\Omega), X)_{[1]}$ and $(f_1,\dots,f_n) \in D_n$. Set $X_i = \operatorname{supp} f_i \ (i \in \mathbb{N}_n)$ and set $x = (U(f_1),\dots,U(f_n)) \in X^n$. For each $\lambda \in X'_{[1]}$ we have

$$\left(\sum_{i=1}^n |\langle U(f_i), \lambda\rangle|^p \right)^{1/p} = \left(\sum_{i=1}^n |\langle f_i, U'(\lambda)\rangle|^p \right)^{1/p} \le \left(\sum_{i=1}^n \|\chi_{X_i} U'(\lambda)\|_p^p \right)^{1/p}$$
$$\le \|U'(\lambda)\|_p \le 1 .$$

Hence $\mu_{p,n}(\lambda) \leq 1$, and so $B_n(X) \subset C_n(X)$.

Conversely, take $x = (x_1, \ldots, x_n) \in C_n(X)$. Choose non-null, pairwise disjoint subsets $X_1, \ldots, X_n \subset \Omega$ with $m(X_i) < \infty$ $(i \in \mathbb{N}_n)$. Set $f_i = \frac{\chi_{X_i}}{m(X_i)^{1/q}}$ $(i \in \mathbb{N}_n)$, so that $(f_1, \ldots, f_n) \in D_n$.

Set

$$U = \sum_{i=1}^{n} x_i \otimes \frac{\chi_{X_i}}{m(X_i)^{1/p}} \in \mathcal{L}(L^q(\Omega), X)\,.$$

For $f \in L^q(\Omega)$, we have

$$\begin{aligned}\|U(f)\| &= \left\| \sum_{i=1}^{n} \left\langle f, \frac{\chi_{X_i}}{m(X_i)^{1/p}} \right\rangle x_i \right\| \leq \mu_{p,n}(x) \left(\sum_{i=1}^{n} \left| \left\langle f, \frac{\chi_{X_i}}{m(X_i)^{1/p}} \right\rangle \right|^q \right)^{1/q} \\ &\leq \mu_{p,n}(x) \left(\sum_{i=1}^{n} \|\chi_{X_i} f\|_q^q \right)^{1/q} \leq \mu_{p,n}(x)\, \|f\|\end{aligned}$$

It follows that $U \in \mathcal{B}(L^q(\Omega), X)_{[1]}$. Since $x = (U(f_1), \ldots, U(f_n))$, we have $B_n(X) = C_n(X)$, as required. □

Let F be a Banach space. Since $L^q(\Omega)$ is reflexive, every $T \in \mathcal{B}(L^q(\Omega), F')$ can be written as $T = U'$ where $U \in \mathcal{B}(F, L^p(\Omega))$. Hence

$$B_n(F') = \left\{ (U'(f_1), \ldots, U'(f_n)) : U \in \mathcal{B}(F, L^p(\Omega))_{[1]},\ (f_1, \ldots, f_n) \in D_n \right\}\,. \tag{4.9}$$

Corollary 4.6.19. *Let F be a Banach space, and let $1 \leq p < \infty$. Let $(\Omega, \mathcal{F}, m)$ be a measure space, such that there exists an infinite collection of pairwise disjoint subsets $\{X_n : n \in \mathbb{N}\} \subset \mathcal{F}$ with $0 < m(X_n) < \infty$ $(n \in \mathbb{N})$. Then*

$$\|(x_1, \ldots, x_n)\|_n^{\mathcal{B}} = \sup \left\{ \left(\sum_{i=1}^{n} |\langle x_i, \lambda_i \rangle|^p \right)^{1/p} : \mu_{p,n}(\lambda) \leq 1 \right\}.$$

Proof. Take $x = (x_1, \ldots, x_n) \in F^n$. We have

$$\begin{aligned}\|x\|_n^{\mathcal{B}} &= \sup \left\{ \left(\sum_{i=1}^{n} |\langle U(x_i), f_i \rangle|^p \right)^{1/p} : U \in \mathcal{B}(F, L^p(\Omega))_{[1]},\ (f_i) \in D_n \right\} \\ &= \sup \left\{ \left(\sum_{i=1}^{n} |\langle x_i, U'(f_i) \rangle|^p \right)^{1/p} : U \in \mathcal{B}(F, L^p(\Omega))_{[1]},\ (f_i) \in D_n \right\} \\ &= \sup \left\{ \left(\sum_{i=1}^{n} |\langle x_i, \lambda_i \rangle|^p \right)^{1/p} : \mu_{p,n}(\lambda) \leq 1 \right\},\end{aligned}$$

where the last line follows from (4.9) and Lemma 4.6.18. □

The following lemma is needed for the applications in the next chapter.

Lemma 4.6.20. *Let Ω be a measure space, and take $1 \leq p < \infty$. Let F be a Banach space. Then we have*

$$\|(\Phi_1, \dots, \Phi_n)\|_n^{\mathcal{B}} = \sup \|(U''(\Phi_1), \dots, U''(\Phi_n))\|_n^{\mathrm{lat}}$$

where $\Phi_1, \dots, \Phi_n \in F''$ and the supremum is taken over all $U \in \mathcal{B}(F, L^{p_1}(\Omega))_{[1]}$.

Proof. Take $\Phi = (\Phi_1, \dots, \Phi_n) \in (F'')^n$. By Corollary 4.6.19 and Lemma 4.6.8 we have

$$\|\Phi\|_n^{\mathcal{B}} = \sup \left\{ \left(\sum_{i=1}^n |\langle \lambda_i, \Phi_i \rangle|^p \right)^{1/p} : \lambda \in (F')^n,\ \mu_{p,n}(\lambda) \leq 1 \right\}.$$

By (4.9) and Lemma 4.6.18 this is equal to

$$\begin{aligned} \|\Phi\|_n^{\mathcal{B}} &= \sup \left\{ \left(\sum_{i=1}^n |\langle U'(f_i), \Phi_i \rangle|^p \right)^{1/p} : U \in \mathcal{B}(F, L^p(\Omega))_{[1]},\ (f_i) \in D_n \right\} \\ &= \sup \left\{ \left(\sum_{i=1}^n |\langle U''(\Phi_i), f_i \rangle|^p \right)^{1/p} : U \in \mathcal{B}(F, L^p(\Omega))_{[1]},\ (f_i) \in D_n \right\} \\ &= \sup \left\{ \|(U''(\Phi_1), \dots, U''(\Phi_n))\|_n^{\mathrm{lat}} : U \in \mathcal{B}(F, L^p(\Omega))_{[1]} \right\}. \end{aligned}$$

□

4.6.6 Multi-norms generated by special-norms

In this section we describe a way of constructing multi-norms over an arbitrary Banach space, with certain 'maximal' properties. We are motivated to answer the following question. The norms $(\|\cdot\|_n^{\mathcal{B}} : n \in \mathbb{N})$ of the previous section, over any Banach space E, satisfy

$$\|(x_1, \dots, x_n)\|_n^{\mathcal{B}} \leq \left(\sum_{i=1}^n \|x_i\| \right)^{1/p} \qquad (x_1, \dots, x_n \in E,\ n \in \mathbb{N}).$$

Does there exist a bigger multi-norm that still satisfies these upper bounds? We shall show that the answer is yes.

It is convenient to have the following rephrasing of the axioms for a multi-normed space to hand. For a set S we define the *diagonal* $\Delta^2(S) = \{(s, s) : s \in S\} \subset S^2$.

Proposition 4.6.21. *Let E be a Banach space, and let $(\|\cdot\|_n : n \in \mathbb{N})$ be a sequence such that $\|\cdot\|_n$ is a norm on E^n $(n \in \mathbb{N})$ with $\|\cdot\|_1 = \|\cdot\|$. Set $B_n = (E^n, \|\cdot\|_n)_{[1]}$ $(n \in \mathbb{N})$. Then the sequence $(\|\cdot\|_n : n \in \mathbb{N})$ is a multi-norm if and only if for each $n \geq 2$ we have:*

(A1) $A_\sigma(B_n) = B_n \quad (\sigma \in \mathfrak{G}_n)$;

(A2) $M_\alpha(B_n) \subset B_n \quad (\alpha \in \overline{\mathbb{D}}^n)$;

(A3) $B_{n-1} \times \{0\} = B_n \cap (E^{n-1} \times \{0\})$;

(A4) $(E^{n-2} \times \Delta^2(E)) \cap (B_{n-1} \times E) \subset (E^{n-2} \times \Delta^2(E)) \cap B_n$. □

Let E be a linear space, and let $n \in \mathbb{N}$. For a set $X \subset E^n$ and $k \in \mathbb{N}$, let

$$\widetilde{X} = \{(x_1, \dots, x_n, y_1, \dots, y_k) : (x_1, \dots, x_n) \in X,\ y_1, \dots, y_k \in \{x_1, \dots, x_n\}\}$$

and define $\nabla^k(X) \subset E^{n+k}$ by

$$\nabla^k(X) = \bigcup_{\alpha \in \overline{\mathbb{D}}^{n+k}} \bigcup_{\sigma \in \mathfrak{S}_{n+k}} M_\alpha(A_\sigma(\widetilde{X}))\,.$$

We also set $\nabla^0(X) = X$. We note the following properties of ∇^k;

(i) $\nabla^k(\langle X \rangle) = \left\langle \nabla^k(X) \right\rangle$;

(ii) $\nabla^k(X \cup Y) = \nabla^k(X) \cup \nabla^k(Y)$;

(iii) $\nabla^k(\nabla^l(X)) = \nabla^{k+l}(X)$.

Let E be a special-normed space. Set $B_n = (E^n, \|\cdot\|_n)_{[1]}$ $(n \in \mathbb{N})$. We inductively define $K_1 = E_{[1]}$ and for each $n \geq 2$ set $K_n = \langle \nabla^1(K_{n-1}) \cup B_n \rangle$. It follows from properties (i), (ii) and (iii) above that

$$K_n = \left\langle \bigcup_{k \in \mathbb{N}_n} \nabla^{n-k}(B_k) \right\rangle \quad (n \in \mathbb{N})\,.$$

The set K_n is absolutely convex and absorbing, hence the Minkowski functional p_n of K_n is a norm on E^n.

Proposition 4.6.22. *Let E be a Banach space, and let $(\|\cdot\|_n : n \in \mathbb{N})$ be a special-norm over E.*

(i) *$(p_n : n \in \mathbb{N})$ is a multi-norm over E, and $p_n(\,\cdot\,) \leq \|\cdot\|_n$ $(n \in \mathbb{N})$.*

(ii) *Let $(|||\cdot|||_n : n \in \mathbb{N})$ be a another multi-norm over E with*

$$|||\cdot|||_n \leq \|\cdot\|_n \quad (n \in \mathbb{N})\,.$$

Then $|||\cdot|||_n \leq p_n(\,\cdot\,)$ $(n \in \mathbb{N})$.

Proof. (i) Let $n \in \mathbb{N}$. Clearly $A_\sigma(K_n) = K_n$ $(\sigma \in \mathfrak{S}_n)$ and $M_\alpha(K_n) \subset K_n$ $(\alpha \in \mathbb{D}^n)$. Hence axioms (A1) and (A2) hold.

Next we shall show that (A3) holds. We first note that if $k \leq n$, and $(x_1, \dots, x_n) \in \nabla^{n-k}(B_k) \subset K_n$, then $(x_1, \dots, x_{n-1}) \in \nabla^{n-k-1}(B_k) \subset K_{n-1}$. Now take $x = (x_1, \dots, x_{n-1}, 0) \in K_n$. We can write x as a convex combination

$$(x_1, \dots, x_{n-1}, 0) = \sum_{i=1}^{N} t_i(y_{1,i}, \dots, y_{n,i})$$

where $(y_{1,i}, \dots, y_{n,i}) \in \bigcup_{k \in \mathbb{N}_n} \nabla^{n-k}(B_k)$ $(i \in \mathbb{N}_N)$. Then

$$(x_1, \dots, x_{n-1}) = \sum_{i=1}^{N} t_i(y_{1,i}, \dots, y_{n-1,i})\,,$$

and $(y_{1,i}, \ldots, y_{n-1,i}) \in K_{n-1}$ $(i \in \mathbb{N}_N)$. Therefore $(x_1, \ldots, x_{n-1}) \in K_{n-1}$. Conversely, take $(x_1, \ldots, x_{n-1}) \in K_{n-1}$. Then $(x_1, \ldots, x_{n-1}, 0) \in K_n$ and so (A3) holds.

Axiom (A4) is immediate from the recursive definition of the sets K_n. Therefore $(p_n : n \in \mathbb{N})$ is a multi-norm. For each $n \in \mathbb{N}$, $B_n \subset K_n$, equivalently $p_n(\,\cdot\,) \leq \|\cdot\|_n$.

(ii) Set $C_n = (E^n, \|\cdot\|_n)_{[1]}$ $(n \in \mathbb{N})$. For each $n \in \mathbb{N}$, $B_n \subset C_n$, which implies that $K_n \subset C_n$, giving the result. □

Definition 4.6.23. Let E be a Banach space, and let $(\|\cdot\|_n : n \in \mathbb{N})$ be a special-norm over E. The *multi-norm generated by* $(\|\cdot\|_n : n \in \mathbb{N})$ is the multi-norm $(d_n : n \in \mathbb{N})$ described above. We denote this multi-normed space by $d(E)$.

Let E be a special-normed space. For $n \in \mathbb{N}$ and $x_1, \ldots, x_n \in E$, we set

$$g_n(x_1, \ldots, x_n) = \sup \left\{ \left\| \left(\sum_{i \in X_1} \alpha_i x_i, \ldots, \sum_{i \in X_m} \alpha_i x_i \right) \right\|_m \right\}$$

where the supremum is taken over all $m \in \mathbb{N}_n$, all partitions $(X_1, \ldots, X_m)$ of $\mathbb{N}_n$ and all complex numbers $\alpha_1, \ldots, \alpha_n \in \overline{\mathbb{D}}$.

Proposition 4.6.24. *Let E be a Banach space, and let $(\|\cdot\|_n : n \in \mathbb{N})$ be a special-norm over E. Then:*

(i) *$(g_n : n \in \mathbb{N})$ is a dual multi-norm over E, and $g_n(\,\cdot\,) \geq \|\cdot\|_n$ $(n \in \mathbb{N})$.*

(ii) *Let $(|||\cdot|||_n : n \in \mathbb{N})$ be another dual multi-normed over E with*

$$|||\cdot|||_n \geq \|\cdot\|_n \quad (n \in \mathbb{N})\,.$$

Then $|||\cdot|||_n \geq g_n(\,\cdot\,)$ $(n \in \mathbb{N})$.

Proof. (i) This follows immediately from the definition.

(ii) Take $n \in \mathbb{N}$ and $x = (x_1, \ldots, x_n) \in E^n$. For any partition $(X_1, \ldots, X_m)$ of $\mathbb{N}_n$ and complex numbers $\alpha_1, \ldots, \alpha_n \in \overline{\mathbb{D}}$, we have

$$\begin{aligned} \left\| \left(\sum_{i \in X_1} \alpha_i x_i, \ldots, \sum_{i \in X_m} \alpha_i x_i \right) \right\|_m &\leq \left|\left|\left| \left(\sum_{i \in X_1} \alpha_i x_i, \ldots, \sum_{i \in X_m} \alpha_i x_i \right) \right|\right|\right|_m \\ &\leq |||(x_1, \ldots, x_n)||| \quad \text{by Lemma 4.6.3 and (A2)}\,. \end{aligned}$$

Therefore $g_n(x) \leq |||x|||_n$. □

Definition 4.6.25. Let E be a Banach space, and let $(\|\cdot\|_n : n \in \mathbb{N})$ be a special-norm over E. The *dual multi-norm generated by* $(\|\cdot\|_n : n \in \mathbb{N})$ is the dual multi-norm $(g_n : n \in \mathbb{N})$ described above. We denote this dual multi-normed space by $g(E)$.

Proposition 4.6.26. *Let E be a special-normed space. Then there is a multi-isometry of dual multi-normed spaces*

$$d(E)^{\times} = g(E^{\times})\,.$$

Hence

$$d_n(x_1,\dots,x_n) = \sup\left\{\left|\sum_{i=1}^{n}\langle x_i,\lambda_i\rangle\right| : \lambda_1,\dots,\lambda_n \in E',\, g_n(\lambda_1,\dots,\lambda_n)\le 1\right\} \qquad (4.10)$$

for each $n\in\mathbb{N}$ and $x_1,\dots,x_n\in E$.

Proof. The family $\{((E')^n, d_n^{\times}) : n\in\mathbb{N}\}$ is a dual multi-normed space, and further, $d_n' \ge \|\cdot\|'$ $(n\in\mathbb{N})$. Hence by Proposition 4.6.24 $d_n' \ge g_n$ $(n\in\mathbb{N})$.

We shall prove the reverse inequality. Take $n\in\mathbb{N}$ and $\lambda=(\lambda_1,\dots,\lambda_n)\in(E')^n$. Let $k\in\mathbb{N}_n$ and take $x\in\nabla^{n-k}(B_k)$. Then

$$x = (x_1,\dots,x_n) = M_\alpha\circ A_\sigma(z_1,\dots,z_k,y_1,\dots,y_{n-k})\,,$$

where $\sigma\in\mathfrak{S}_n$, $\alpha=(\alpha_i)\in\overline{\mathbb{D}}^n$, $(z_1,\dots,z_k)\in B_k$, and $\{y_1,\dots,y_{n-k}\}\subset\{z_1,\dots,z_k\}$. Set

$$X_j = \{i\in\mathbb{N}_n : x_i = z_j\} \quad (j\in\mathbb{N}_k)\,.$$

The sets $X_1,\dots,X_k$ form a partition of $\mathbb{N}_n$ and we have

$$|\langle x,\lambda\rangle| = \left|\sum_{j=1}^{k}\left\langle z_j, \sum_{i\in X_j}\alpha_i\lambda_i\right\rangle\right| \le \left\|\left(\sum_{i\in X_1}\alpha_i\lambda_i,\dots,\sum_{i\in X_k}\alpha_i\lambda_i\right)\right\|_m' \le g_n(\lambda)\,.$$

Hence $d_n^{\times}(\lambda)\le g_n(\lambda)$ as required.

The identity (4.10) follows from the Hahn–Banach theorem. □

Example 4.6.27. Let Ω be a measure space, and set $E=L^1(\Omega)$. Take $1\le p<\infty$, and regard E as a special normed space equipped with the family of p-summing norms $(\|\cdot\|_{\ell_n^p} : n\in\mathbb{N})$. The special normed space $E^{\times}$ is the Banach space E' equipped with the family $(\|\cdot\|_{\ell_n^q} : n\in\mathbb{N})$, where q is the conjugate index to p. The dual multi-norm over E' generated by $(\|\cdot\|_{\ell_n^q} : n\in\mathbb{N})$ is given by

$$g_n(\lambda) = \max_{\mathbf{X}}\left\{\left\|\left(\sum_{i\in X_1}|\lambda_i|,\dots,\sum_{i\in X_m}|\lambda_i|\right)\right\|_{\ell_m^q}\right\},$$

where $\lambda=(\lambda_1,\dots,\lambda_n)\in L^\infty(\Omega)^n$ and the maximum is taken over all partitions $\mathbf{X}=(X_1,\dots,X_m)$ of $\mathbb{N}_n$.

Now take $f = (f_1, \dots, f_n) \in L^1(\Omega)^n$, $\lambda = (\lambda_1, \dots, \lambda_n) \in L^\infty(\Omega)^n$, and a partition $(X_1, \dots, X_m)$ of $\mathbb{N}_n$. We have

$$\begin{aligned}
|\langle f, \lambda\rangle| &\leq \sum_{k=1}^{m} \left(\sum_{i\in X_k} |\langle f_i, \lambda_i\rangle| \right) \leq \sum_{k=1}^{m} \left(\left\langle \max_{i\in X_k} |f_i|\,, \sum_{i\in X_k} |\lambda_i| \right\rangle \right) \\
&\leq \left\| \left(\max_{i\in X_1} |f_i|\,, \dots, \max_{i\in X_m} |f_i| \right) \right\|_{\ell^p_m} \left\| \left(\sum_{i\in X_1} |\lambda_i|\,, \dots, \sum_{i\in X_m} |\lambda_i| \right) \right\|_{\ell^q_m} \\
&\leq \left\| \left(\max_{i\in X_1} |f_i|\,, \dots, \max_{i\in X_m} |f_i| \right) \right\|_{\ell^p_m} g_n(\lambda_1, \dots, \lambda_n)\,.
\end{aligned}$$

It follows that the multi-norm over E generated by $(\|\cdot\|_{\ell^p_n} : n \in \mathbb{N})$ satisfies

$$d_n(f_1, \dots, f_n) \leq \min_{\mathbf{X}} \left\{ \left\| \left(\max_{i\in X_1} |f_i|\,, \dots, \max_{i\in X_m} |f_i| \right) \right\|_{\ell^p_m} \right\},$$

where the minimum is taken over all partitions $\mathbf{X} = (X_1, \dots, X_m)$ of $\mathbb{N}_n$. By Proposition 4.6.22 we have

$$\begin{aligned}
d_n(f_1, \dots, f_n) &= \min_{\mathbf{X}} \left\{ \left\| \left(\max_{i\in X_1} |f_i|\,, \dots, \max_{i\in X_m} |f_i| \right) \right\|_{\ell^p_m} \right\} \\
&= \min_{\mathbf{X}} \left(\sum_{j=1}^{m} \left\| \max_{i\in X_j} |f_i| \right\|^p \right)^{1/p}.
\end{aligned}$$

Chapter 5

Injectivity of the $L^1(G)$-module $L^p(G)$

Let G be a locally compact group, and let $1 < p < \infty$. Since $L^p(G)$ is a dual Banach $L^1(G)$-module, it follows from Johnson's theorem and Proposition 1.3.11 that $L^p(G)$ is an injective left $L^1(G)$-module whenever G is amenable. In [8] the authors obtained a partial converse in the case where G is discrete. They showed in [8, Theorem 5.12] that, if G is a discrete group and $\ell^p(G)$ is injective for some $p \in (1, \infty)$, then G must be 'pseudo-amenable'. In subsequent work the same authors used the concept of a multi-norm to define another generalized notion of amenability, called *p-amenability.* They showed that, if $\ell^p(G)$ is injective, then G must be p-amenable.

In this chapter, we define another generalized notion of amenability, called *super p-amenability.* We prove for a discrete group G, that $\ell^p(G)$ is injective if and only if G is super p-amenable. For a general locally compact group G we show the following implications

$$L^p(G) \text{ injective} \Rightarrow G \text{ super } p\text{-amenable} \Rightarrow G \ p\text{-amenable} \ \Rightarrow G \text{ pseudo-amenable}\,.$$

5.1 Generalized notions of amenability

Definition 5.1.1 ([7, Definition 6.3])**.** Let E be a multi-normed space. A subset $B \subset E$ is *multi-bounded* if

$$c_B := \sup \left\{ \|(x_1, \ldots, x_n)\|_n : x_1, \ldots, x_n \in B,\, n \in \mathbb{N} \right\} < \infty\,.$$

Let E, F be multi-normed spaces, and let $T \in \mathcal{M}(E, F)$. It follows immediately from the definitions that $T(B)$ is a multi-bounded set in F whenever B is a multi-bounded set in E. Conversely, it is proved in [7, Proposition 6.10] that any $T \in$

$\mathcal{B}(E,F)$ which takes multi-bounded sets to multi-bounded sets is multi-bounded, and further

$$\|T\|_{mb} = \sup\left\{c_{T(B)} : c_B \le 1\right\}.$$

Definition 5.1.2. Let G be a locally compact group, and take $1 \le p < \infty$. A mean $\Lambda \in L^1(G)''$ is *left p-invariant* if the set $\{s \cdot \Lambda : s \in G\}$ is multi-bounded in the standard p-multi-norm on $L^1(G)''$. If there exists such an element $\Lambda \in L^1(G)''$, then G is *p-amenable*.

By replacing the standard p-multi-norm on $L^1(G)''$ by the extension of the lattice multi-norm on $L^p(G)$, $(\|\cdot\|_n^{\mathcal{B}} : n \in \mathbb{N})$, then we obtain the definition of a *left super p-invariant mean* and of a *super p-amenable group*.

The idea behind this definition is an attempt to measure the 'left-invariance' of a mean $\Lambda \in L^1(G)''$ by measuring the growth of the sets $\{s \cdot \Lambda : s \in F\}$ as F ranges through all finite subsets of G.

For any Banach space E and $n \in \mathbb{N}$, we have

$$\{x \in E^n : \mu_{1,n}(x) \le 1\} \subset \{x \in E^n : \mu_{p,n}(x) \le 1\}.$$

Hence by Corollary 4.6.19 we have

$$\|(\Phi_1,\dots,\Phi_n)\|_n^{(1,p)} \le \|(\Phi_1,\dots,\Phi_n)\|_n^{\mathcal{B}} \quad (\Phi_1,\dots,\Phi_n \in L^1(G)'').$$

It is clear from this inequality that super p-amenability is stronger than p-amenability.

There is an obvious definition of a *right p-invariant mean* and of a *right super p-invariant mean*. Set $A = L^1(G)$. Define $T : A \to A$ by

$$T(a)(s) = a(s^{-1})\Delta(s^{-1}) \quad (a \in A, s \in G).$$

Then $T'' : A'' \to A''$ with and $T'(1) = 1$, and for each $\Lambda \in L^1(G)''$, T takes the set $\{s \cdot \Lambda : s \in G\}$ to $\{T''(\Lambda) \cdot s : s \in G\}$. Since $T'' \in \mathcal{M}(A'')$ when A'' is equipped with either the standard p-multi-norm or the multi-norm $\{\|\cdot\|_n^{\mathcal{B}} : n \in \mathbb{N}\}$, it follows that there exists a left [super] p-invariant mean if and only if there exists a right [super] p-invariant mean.

5.1.1 p-amenability

Let G be a locally compact group, and take $1 \le p < \infty$. Let $\Lambda \in L^1(G)''$ be a p-invariant mean. We set

$$c_\Lambda = \sup\{\|(s_1 \cdot \Lambda,\dots,s_n \cdot \Lambda)\|_n^{(1,p)} : s_1,\dots,s_n \in G,\, n \in \mathbb{N}\}.$$

Proposition 5.1.3. *Let G be a locally compact group, and take $1 \le p < \infty$. Then there exists a p-invariant mean in $L^1(G)$ if and only if G is compact.*

Proof. If G is compact then $\Lambda = \chi_G/m(G)$ is an invariant and hence p-invariant mean.

Conversely, assume towards a contradiction that G is not compact and that there exists a p-invariant mean $a \in L^1(G)$. There is a compact set V such that $c = \int_V |a(t)| \, \mathrm{d}m(t) \neq 0$. Since G is not compact, for each $N \in \mathbb{N}$, there exist elements $s_1, \ldots, s_N \in G$ such that the sets $s_1V, \ldots, s_NV$ are pairwise disjoint. We have $\chi_{s_iV}(s_i \cdot a) = s_i \cdot (\chi_V a)$ and so

$$c_a^p \geq \|(s_1 \cdot a, \cdots, s_n \cdot a)\|_n^{(1,p)} \geq \sum_{i=1}^{N} \|\chi_{s_iV}(s_i \cdot a)\|^p = \sum_{i=1}^{N} \|\chi_V a\|^p = Nc^p \,.$$

This holds for all $N \in \mathbb{N}$, the required contradiction. □

The following result was first proved by Dales and Polyakov for discrete groups.

Theorem 5.1.4. *Let G be a locally compact group. Then G is 1-amenable if and only if G is amenable.*

Proof. It is clear that every amenable locally compact group is 1-amenable.

We set $A = L^1(G)$. Suppose that G is 1-amenable, and let $\Lambda \in L^1(G)''$ be a 1-invariant mean. For each $U \in \mathcal{B}_G$ we define $\left\langle \chi_U, \widetilde{\Lambda} \right\rangle \in \mathbb{R}^+$ by

$$\left\langle \chi_U, \widetilde{\Lambda} \right\rangle = \sup \sum_{i=1}^{n} \langle \chi_{X_i}, s_i \cdot \Lambda \rangle \,,$$

where the supremum is taken over all $n \in \mathbb{N}$, all $s_1, \ldots, s_n \in G$, and all measurable partitions $(X_1, \ldots, X_n)$ of U. The supremum is finite since Λ is 1-invariant. Since

$$\left\langle U \cup V, \widetilde{\Lambda} \right\rangle = \left\langle U, \widetilde{\Lambda} \right\rangle + \left\langle V, \widetilde{\Lambda} \right\rangle \quad (U, V \in \mathcal{B}_G)\,,$$

we can extend Λ to a linear map $\widetilde{\Lambda} : \mathcal{S} = \mathrm{lin}\,\{\chi_U : U \in \mathcal{B}_G\} \to \mathbb{C}$ by setting

$$\left\langle \sum_{i=1}^{n} \alpha_i \chi_{U_i}, \widetilde{\Lambda} \right\rangle = \sum_{i=1}^{n} \alpha_i \left\langle \chi_{U_i}, \widetilde{\Lambda} \right\rangle \,.$$

The set $\mathcal{S}$ is dense in A', and $\left|\left\langle \lambda, \widetilde{\Lambda} \right\rangle\right| \leq c_\Lambda \|\lambda\|_\infty$ $(\lambda \in \mathcal{S})$. Hence $\widetilde{\Lambda}$ extends to an element $\widetilde{\Lambda} \in A''$ with $\left\|\widetilde{\Lambda}\right\| \leq c_\Lambda$. It is easily checked that $\left\langle 1, \widetilde{\Lambda} \right\rangle = 1$ and $s \cdot \widetilde{\Lambda} = \widetilde{\Lambda}$ $(s \in G)$. This implies that G is amenable. □

Pseudo-amenability

Here we show that p-amenability implies pseudo-amenability. In the next proposition we use the estimate

$$\|(\Phi_1, \ldots, \Phi_n)\|_n^{(1,1)} \leq n^{1/q} \|(\Phi_1, \ldots, \Phi_n)\|_n^{(1,p)} \quad (\Phi_1, \ldots, \Phi_n \in L^1(G'')) \,.$$

Proposition 5.1.5. *Let G be a locally compact group, and take $1 < p < \infty$. Suppose that G is p-amenable. Then there exists $C \geq 1$ such that, for each $n \in \mathbb{N}$, and for each finite set $\{s_1, \ldots, s_n\} \subset G$, there exists $a \in P(G)$ with*

$$\|(s_1 \cdot a, \ldots, s_n \cdot a)\|_n^{(1,1)} \leq C n^{1/q}.$$

Proof. We set $A = L^1(G)$.

Let $\Lambda \in A''$ be a p-invariant mean. Set $C = c_\Lambda + 1$. Fix $n \in \mathbb{N}$ and a finite set $\{s_1, \ldots, s_n\} \subset G$. By [29, Proposition (0.1)] there is a net (a_α) in $P(G)$ such that $\lim_\alpha a_\alpha = \Lambda$ in the weak-$*$ topology on A''. Also there is a net $b_\alpha = (b_{1,\alpha}, \ldots, b_{n,\alpha})$ in A^n such that

$$\lim_\alpha b_\alpha = (s_1 \cdot \Lambda, \ldots, s_n \cdot \Lambda)$$

in the weak-$*$ topology on $(A'')^n$ and

$$\sup_\alpha \|b_\alpha\|_n^{(1,1)} \leq \|(s_1 \cdot \Lambda, \ldots, s_n \cdot \Lambda)\|_n^{(1,1)} \leq c_\Lambda n^{1/q}.$$

We can suppose that these nets are indexed by the same directed set. We have

$$\lim_\alpha (s_1 \cdot a_\alpha - b_{1,\alpha}, \ldots, s_1 \cdot a_\alpha - b_{n,\alpha}) = 0$$

in the weak topology on A^n. By Mazur's theorem there is some convex combination

$$\begin{aligned} v &= \sum_{j=1}^N t_j (s_1 \cdot a_{\alpha_j} - b_{1,\alpha_j}, \ldots, s_n \cdot a_{\alpha_j} - b_{n,\alpha_j}) \\ &= \left(s_1 \cdot \left(\sum_{j=1}^N t_j a_{\alpha_j} \right) - \sum_{j=1}^N t_j b_{1,\alpha_j}, \ldots, s_n \cdot \left(\sum_{j=1}^N t_j a_{\alpha_j} \right) - \sum_{j=1}^N t_j b_{n,\alpha_j} \right) \end{aligned}$$

such that $\|v\|_n^{(1,1)} \leq 1$. Set $a = \sum_j t_j a_{\alpha_j}$ and $b_i = \sum_j t_j b_{i,\alpha_j}$ $(i \in \mathbb{N}_n)$.

Then we have

$$\begin{aligned} \|(s_1 \cdot a, \ldots, s_n \cdot a)\|_n^{(1,1)} &\leq \|v\|_n^{(1,1)} + \|(b_1, \ldots, b_n)\|_n^{(1,1)} \\ &\leq 1 + c_\Lambda n^{1/q} \leq C n^{1/q}. \end{aligned}$$

□

Let S be a set, and let $n \in \mathbb{N}$. Then $\mathcal{P}_n(S)$ denotes the collection of subsets of S containing n elements.

Lemma 5.1.6. *Let $f = \sum_{k=1}^N \beta_k \chi_{S_k} \in P(G)$ where $S_1 \subset S_2 \subset \cdots \subset S_N \subset G$. Let $F = \{s_1, \ldots, s_n\} \in \mathcal{P}_n(G)$. Then*

$$\|(s_1 \cdot f, \ldots, s_n \cdot f)\|_n^{(1,1)} = \sum_{k=1}^N |\beta_k| \, m(F S_k).$$

Proof. For $t \in G$ and $i \in \mathbb{N}_n$ we define $k(t,i) \in \mathbb{N}_N$ by

$$k(t,i) = \min\{k \in \mathbb{N}_N : t \in s_i S_k\} .$$

Also for $t \in G$ we define $k(t) \in \mathbb{N}_N$ by

$$k(t) = \min\{k(t,i) : i \in \mathbb{N}_n\} \quad (t \in G) .$$

Now we have

$$(s_i \cdot f)(t) = \sum_{k=1}^{n} \beta_k \chi_{s_i S_k}(t) = \sum_{k=k(t,i)}^{N} \beta_k \quad (i \in \mathbb{N}_n,\, t \in G)$$

and

$$\max_{i \in \mathbb{N}_n} |s_i \cdot f|(t) = \sum_{k=k(t)}^{N} |\beta_k| = \sum_{k=1}^{N} |\beta_k| \, \chi\{t : k(t) \le k\} \quad (t \in G) .$$

Now, for each $k \in \mathbb{N}_N$, we find that

$$\begin{aligned} \{t : k(t) \le k\} &= \{t : \exists i \in \mathbb{N}_n,\, k(t,i) \le k\} \\ &= \{t : \exists i \in \mathbb{N}_n, \exists l \le k,\, t \in s_i S_l\} \\ &= FS_k . \end{aligned}$$

Hence

$$\|(s_1 \cdot f, \ldots, s_n \cdot f)\|_n^{(1,1)} = \int_G \max_{i \in \mathbb{N}_n} |s_i \cdot f|(t)\, \mathrm{d}m(t) = \sum_{k=1}^{N} |\beta_k| \, m(FS_k) . \qquad \square$$

Next we show that the element $a \in P(G)$ given in Proposition 5.1.5 can be taken to be of the form $a = \chi_S / m(S)$ for some non-null compact set S. For a discrete group G this condition is the same as the condition arrived at in [8, Proposition 5.11].

Proposition 5.1.7. *Let G be a locally compact group, and take $1 < p < \infty$. Suppose that G is p-amenable. Then there exists $C \ge 1$ such that, for every $n \in \mathbb{N}$ and every $F \in \mathcal{P}_n(G)$, there exists a non-null, compact subset $S \subset G$ with*

$$\frac{m(FS)}{m(S)} \le C n^{1/q} .$$

Proof. We set $A = L^1(G)$. Let C_0 be the constant given in Proposition 5.1.5. Take $n \in \mathbb{N}$ and $F = \{s_1, \ldots, s_n\} \in \mathcal{P}_n(G)$. By Proposition 5.1.5 there exists $a \in P(G)$ with $\|(s_1 \cdot a, \ldots, s_n \cdot a)\|_n^{(1,1)} \le C_0 n^{1/q}$.

There exists $f \in P(G)$ with finite range, such that $\|f - a\| \le n^{1/q - 1}$. Then we have

$$\begin{aligned} \|(s_1 \cdot f, \ldots, s_n \cdot f)\|_n^{(1,1)} &\le \|(s_1 \cdot a, \ldots, s_n \cdot a)\|_n^{(1,1)} + \|(s_1 \cdot (f-a), \ldots, s_n \cdot (f-a))\|_n^{(1,1)} \\ &\le C_0 n^{1/q} + \sum_{i=1}^{n} \|s_i \cdot (f-a)\| \le (C_0 + 1) n^{1/q} . \end{aligned}$$

Set $C = C_0 + 1$. We can write

$$f = \sum_{k=1}^{N} \alpha_k \chi_{S_k}/m(S_k)\,,$$

where $S_1 \subset S_2 \subset \cdots \subset S_N \subset G$, where $\alpha_1, \ldots, \alpha_N > 0$, and where $\sum_{k=1}^{N} \alpha_k = 1$. By Lemma 5.1.6 we have

$$\sum_{k=1}^{N} \frac{\alpha_k m(FS_k)}{m(S_k)} \le Cn^{1/q}\,.$$

The left-hand side is a convex sum, and hence there exists $k \in \mathbb{N}_N$ such that

$$\frac{m(FS_k)}{m(S_k)} \le Cn^{1/q}\,.$$

Finally we set $S = S_k$, giving the result. □

A discrete group G satisfying the condition in the next proposition is called *pseudo-amenable* in [8, Definition 5.5].

Proposition 5.1.8. *Let G be a locally compact group, and take $1 < p < \infty$. Suppose that G is p-amenable. Then for all $\varepsilon > 0$, there exists $n_0 = n_0(\varepsilon) \in \mathbb{N}$ such that, for all $n \ge n_0$ and $F \in \mathcal{P}_n(G)$, there exists a non-null, compact subset $S \subset G$ with*

$$\frac{m(FS)}{m(S)} \le \varepsilon n\,.$$

Proof. Let $C \ge 1$ be the constant prescribed in Proposition 5.1.7. Take $\varepsilon > 0$, and choose $n_0 \in \mathbb{N}$ with $n_0^{1/p} = n_0^{1-1/q} \ge C/\varepsilon$, so that $Cn^{1/q} \le \varepsilon n \;\; (n \ge n_0)$. Now take $n \in \mathbb{N}$ and $F \in \mathcal{P}_n(G)$. By Proposition 5.1.7 there exists a non-null, compact subset $S \subset G$ with $\frac{m(FS)}{m(S)} \le Cn^{1/q}$. Hence

$$\frac{m(FS)}{m(S)} \le n\varepsilon\,,$$

as required. □

Remark 5.1.9. The following facts are proved in [8]:

(i) Every subgroup of a pseudo-amenable discrete group is pseudo-amenable.

(ii) The free group on 2 generators, $\mathbb{F}_2$ is not pseudo-amenable.

5.2 Injectivity of the $L^1(G)$-module $L^p(G)$

5.2.1 A coretraction problem

Let G be a locally compact group, and take $1 < p < \infty$. We set $A = L^1(G)$, $E = L^p(G)$ and $J = \mathcal{B}(A, E)$. We now define an action of G on the space J by

$$(t * U)\,(a) = t \cdot U(t^{-1} \cdot a) \quad (a \in A)\,. \tag{5.1}$$

For each $U \in J$ and $a \in A$, the map $t \mapsto (t * U)(a) = t \cdot U(t^{-1} \cdot a)$, $G \to E$ is continuous. This follows from the inequality

$$\begin{aligned}\left\|t \cdot U(t^{-1} \cdot a) - U(a)\right\|_p &\leq \left\|t \cdot U(t^{-1} \cdot a) - t \cdot U(a)\right\|_p + \|t \cdot U(a) - U(a)\|_p \\ &= \left\|U(t^{-1} \cdot a - a)\right\|_p + \|t \cdot U(a) - U(a)\|_p \\ &\leq \|U\| \left\|t^{-1} \cdot a - a\right\|_1 + \|t \cdot U(a) - U(a)\|_p \,,\end{aligned}$$

and [4, 3.3.11].

Proposition 5.2.1. *There is a Banach left A-module structure on J given by*

$$(b * U)(a) = \int_G b(t)\,(t * U)(a)\,\mathrm{d}m(t) \quad (a, b \in A,\, U \in J)\,. \tag{5.2}$$

Proof. This is similar to the standard proof that $L^p(G)$ is a left $L^1(G)$-module [4, 3.3.19]. Fix $U \in J$ and $a, b \in A$. Let $\psi \in C_{00}(G)$. By Hölder's inequality, we have

$$\int_G \left|U(t^{-1} \cdot a)(t^{-1}s)\right| |\psi(s)|\,\mathrm{d}m(s) \leq \left\|t \cdot U(t^{-1} \cdot a)\right\|_p \|\psi\|_q \leq \|U\|\,\|a\|_1\,\|\psi\|_q$$

for each $t \in G$. Now define

$$\begin{aligned}\Lambda : C_{00}(G) \to \mathbb{C} : \psi \mapsto &\int_G (b * U)(a)(s)\psi(s)\,\mathrm{d}m(s) \\ &= \int_G \left(\int_G b(t)U(t^{-1} \cdot a)(t^{-1}s)\,\mathrm{d}m(t)\right)\psi(s)\,\mathrm{d}m(s) \\ &= \int_G b(t)\left(\int_G U(t^{-1} \cdot a)(t^{-1}s)\psi(s)\,\mathrm{d}m(s)\right)\mathrm{d}m(t)\,.\end{aligned}$$

Then $|\Lambda(\psi)| \leq \|b\|_1\,\|U\|\,\|a\|_1\,\|\psi\|_q$ $(\psi \in C_{00})$, and so Λ extends to an element of $L^q(G)'$ of norm at most $\|b\|_1\,\|U\|\,\|a\|_1$. Hence by the identification of $L^q(G)'$ with $L^p(G)$, we have $(b * U)(a) \in L^p(G)$ and $b * U \in J$ with $\|b * U\| \leq \|b\|_1\,\|U\|$.

Associativity of $*$ follows in the same way as [23, Proposition 2.1]. □

We shall denote this left A-module by $\widetilde{J} = (J, *)$. (We could similarly define a right multiplication such that J becomes an $L^1(G)$-bimodule).

For a measurable subset $V \subset G$ and $U \in J$ we define $\chi_V U \in J$ by

$$(\chi_V U)(a)(s) = \chi_V(s)U(a)(s) \quad (a \in A,\, s \in G)\,.$$

Now define an embedding $\widetilde{\Pi} : E \to \widetilde{J}$, by

$$(\widetilde{\Pi}x)(a) = \varphi_G(a)x \quad (a \in A)\,.$$

For $b \in A$, we have

$$\left(b * \widetilde{\Pi}x\right)(a) = \int_G b(t)\varphi_G(t^{-1} \cdot a)t \cdot x\,\mathrm{d}m(t) = \varphi_G(a)b \star_p x = \widetilde{\Pi}(b \star_p x)(a) \quad (a \in A)\,,$$

and so $\widetilde{\Pi} \in {}_A\mathcal{B}(E, \widetilde{J})$ is a left A-module morphism. This morphism is admissible (a splitting operator is $U \mapsto U(a_0)$ for any $a_0 \in A$ with $\varphi_G(a_0) = 1$).

Proposition 5.2.2. *Let G be a locally compact group, and let $1 < p < \infty$. Then $L^p(G)$ is injective in $L^1(G)$-**mod** if and only if the morphism $\widetilde{\Pi}$ is a coretraction in $L^1(G)$-**mod**.*

Proof. The condition is necessary by [19, VII.1.34].

As above we set $A = L^1(G)$, $E = L^p(G)$, and $J = \mathcal{B}(A, E)$. Also set $F = L^q(G)$.

Suppose that $\widetilde{\Pi}$ is a coretraction, so that there exists $R \in {}_A\mathcal{B}(\widetilde{J}, E)$ with $R \circ \widetilde{\Pi} = I_E$. For $f \in A$ we define $Q_f \in \mathcal{B}(J)$ by

$$Q_f(U)(a) = (a * U)(f) \quad (a \in A,\, U \in J)\,.$$

For $x \in E$ and $a \in A$, we have

$$Q_f\left(\Pi(x)\right)(a) = (a * \Pi(x))(f) = \varphi_G(a)(f \cdot x) = \left(\widetilde{\Pi}(f \cdot x)\right)(a)\,, \tag{5.3}$$

and for $U \in J$, and $a, b \in A$ we have

$$\begin{aligned}
(b * Q_f(U))(a) &= \int_G b(s) s \cdot Q_f(U)(s^{-1} \cdot a)\, \mathrm{d}m(s) \\
&= \int_G b(s) s \cdot ((s^{-1} \cdot a) * U)(f)\, \mathrm{d}m(s) \\
&= \int_G b(s) s \cdot \left(\int_G a(st) t \cdot U(t^{-1} \cdot f)\, \mathrm{d}m(t) \right) \mathrm{d}m(s) \\
&= \int_G b(s) s \cdot \left(\int_G a(t) s^{-1} t \cdot U(t^{-1} s \cdot f)\, \mathrm{d}m(t) \right) \mathrm{d}m(s) \\
&= \int_G b(s) \left(\int_G a(t) t \cdot U(t^{-1} s \cdot f)\, \mathrm{d}m(t) \right) \mathrm{d}m(s) \quad \text{(by 1.1.10)} \\
&= \int_G a(t) \left(\int_G b(s) t \cdot U(t^{-1} s \cdot f)\, \mathrm{d}m(s) \right) \mathrm{d}m(t) \quad \text{(by Fubini)} \\
&= \int_G a(t) t \cdot U \left(\int_G b(s) t^{-1} s \cdot f\, \mathrm{d}m(s) \right) \mathrm{d}m(t) \quad \text{(by 1.1.10)} \\
&= \int_G a(t) t \cdot U(t^{-1} \cdot b \star f)\, \mathrm{d}m(t) \quad \text{(by 1.5.1)} \\
&= (a * U)(b \star f)\,.
\end{aligned}$$

Hence

$$(b * Q_f(U))(a) = (a * U)(b \star f)\,. \tag{5.4}$$

We also have

$$Q_f(b \cdot U)(a) = \int_G a(t) t \cdot U(t^{-1} \cdot f \star b) = (a * U)(f \star b)\,. \tag{5.5}$$

Let (e_α) be a bounded approximate identity for A, and set $Q_\alpha = Q_{e_\alpha}$. Let Q be a weak-$*$ cluster point in $\mathcal{B}(J, J) = \left(J \widehat{\otimes} (A \widehat{\otimes} F)\right)'$ of the bounded net (Q_α). By passing to a subnet we may suppose that $Q = \lim_\alpha Q_\alpha$ in the weak-$*$ topology. Take $x \in E$. Then for each $a \in A$ and $\lambda \in F$, by (5.3) we have

$$\langle \lambda, Q(\Pi x)(a) \rangle = \lim_\alpha \left\langle \lambda, \left(\widetilde{\Pi}(e_\alpha \cdot x)\right)(a) \right\rangle = \left\langle \lambda, \left(\widetilde{\Pi} x\right)(a) \right\rangle\,.$$

Hence $Q \circ \Pi = \widetilde{\Pi}$. Take $U \in J$ and $b \in A$. Then for each $a \in A$, and $\lambda \in F$, by (5.4) and (5.5) we have

$$\langle \lambda, (b * Q(U))(a) \rangle = \lim_\alpha \langle \lambda, (a * U)(b \star e_\alpha) \rangle = \lim_\alpha \langle \lambda, (a * U)(e_\alpha \star b) \rangle$$
$$= \lim_\alpha \langle \lambda, Q_\alpha(b \cdot U)(a) \rangle = \langle \lambda, Q(b \cdot U) \rangle \, .$$

Hence $b * Q(U) = Q(b \cdot U)$ and $Q \in {}_A\mathcal{B}(J, \widetilde{J})$.

Finally we set $\rho = R \circ Q$, then $\rho \in {}_A\mathcal{B}(J, E)$ and $\rho \circ \Pi = I_E$. Therefore E is injective in A-**mod**. □

5.2.2 Main result

Let G be a locally compact group, and let $1 < p < \infty$. We shall prove that, if $L^p(G)$ is injective in $L^1(G)$-**mod**, then G must be super p-amenable.

We start with a generalization of [8, Lemma 5.2]. For $n \in \mathbb{N}$, we set $D_n = \{-1, 1\}^n$, and for $j \in \mathbb{N}_n$ we set

$$D_n^+(j) = \{(d_1, \ldots, d_n) \in D_n : d_j = 1\}, \quad D_n^-(j) = \{(d_1, \ldots, d_n) \in D_n : d_j = -1\} \, .$$

Lemma 5.2.3. *Let $n \in \mathbb{N}$, let E be a normed space, and let $F : \mathbb{N}_n \times \mathbb{N}_n \to E$. Set*

$$C = \max \left\{ \left(\sum_{j=1}^n \left\| \sum_{i=1}^n d_i F(i,j) \right\|^p \right)^{1/p} : (d_1, \ldots, d_n) \in D_n \right\} .$$

Then

$$\left(\sum_{j=1}^n \|F(j,j)\|^p \right)^{1/p} \leq C \, .$$

Proof. Take $d = (d_1, \ldots, d_n) \in D_n$, and set $x_{j,d} = \sum_{i=1}^n d_i F(i,j)$ $(j \in \mathbb{N}_n)$. By hypothesis, we have $\sum_{j=1}^n \|x_{j,d}\|^p \leq C^p$. Since there are 2^n elements in D_n, we have

$$\sum_{j=1}^n \sum_{d \in D_n} \|x_{j,d}\|^p \leq 2^n C^p \, .$$

For each $j \in \mathbb{N}_n$ we can write the term $\sum_{d \in D_n} \|x_{j,d}\|^p$ as

$$\begin{aligned}
\sum_{d \in D_n} \|x_{j,d}\|^p &= \sum_{d \in D_n^+(j)} \|x_{j,d}\|^p + \sum_{d \in D_n^-(j)} \|x_{j,d}\|^p \\
&= \sum_{d \in D_{n-1}} \left\| \sum_{i \neq j} d_i F(i,j) + F(j,j) \right\|^p + \left\| \sum_{i \neq j} d_i F(i,j) - F(j,j) \right\|^p \\
&\geq \sum_{d \in D_{n-1}} 2 \|F(j,j)\|^p \quad (\text{by Jensen's inequality}) \\
&= 2^{n-1}.2 \|F(j,j)\|^p = 2^n \|F(j,j)\|^p \, .
\end{aligned}$$

This holds for each $j \in \mathbb{N}_n$, and so summing over j we get

$$2^n \sum_{j=1}^{n} \|F(j,j)\|^p \leq \sum_{j=1}^{n} \sum_{d \in D_n} \|x_{j,d}\|^p \leq 2^n C^p .$$

Hence we have $\sum_{j=1}^{n} \|F(j,j)\|^p \leq C^p$, and the result follows. □

Proposition 5.2.4. *Let Ω be a measure space, and take p with $1 < p < \infty$. Let $R : \mathcal{B}(L^1(\Omega), L^p(\Omega)) \to L^p(\Omega)$ be a bounded linear operator, and let $(X_i)_{i=1}^n$ and $(Y_i)_{i=1}^n$ be measurable partitions of Ω. Then*

$$\left(\sum_{i=1}^{n} \|\chi_{X_i} R(\chi_{Y_i} U)\|_p^p \right)^{1/p} \leq \|R\| \, \|U\| \quad (U \in \mathcal{B}(L^1(\Omega), L^p(\Omega))) .$$

Proof. We define $F : \mathbb{N}_n \times \mathbb{N}_n \to L^p(\Omega)$ by

$$F(i,j) = \chi_{X_j} R(\chi_{Y_i} U) \quad (i,j \in \mathbb{N}_n) .$$

For each $(d_1, \ldots, d_n) \in D_n$ we have

$$\begin{aligned} \sum_{j=1}^{n} \left\| \sum_{i=1}^{n} d_i F(i,j) \right\|^p &= \sum_{j=1}^{n} \left\| \sum_{i=1}^{n} d_i \chi_{X_j} R(\chi_{Y_i} U) \right\|^p = \sum_{j=1}^{n} \left\| \chi_{X_j} R \left(\sum_{i=1}^{n} d_i \chi_{Y_i} U \right) \right\|^p \\ &= \left\| R \left(\sum_{i=1}^{n} d_i \chi_{Y_i} U \right) \right\|^p \leq \|R\|^p \left\| \sum_{i=1}^{n} d_i \chi_{Y_i} U \right\|^p = \|R\|^p \, \|U\|^p . \end{aligned}$$

Hence by Lemma 5.2.3 we have

$$\left(\sum_{j=1}^{n} \|F(j,j)\|^p \right)^{1/p} = \left(\sum_{j=1}^{n} \|\chi_{X_j} R(\chi_{Y_j} U)\|^p \right)^{1/p} \leq \|R\| \, \|U\| .$$

□

Lemma 5.2.5. *Let $U \in \mathcal{B}(L^1(\Omega), L^p(\Omega))$, let $f_1, \ldots, f_n \in L^q(\Omega)$ have pairwise disjoint supports, and let $x_1, \ldots, x_n \in L^p(\Omega)$ have pairwise disjoint supports. Set*

$$T = \sum_{i=1}^{n} x_i \otimes U'(f_i) \in \mathcal{L}(L^1(\Omega), L^p(\Omega)) .$$

Then $T \in \mathcal{B}(L^1(\Omega), L^p(\Omega))$ and $\|T\| \leq \|U\| \max\{\|f_i\|_q \|x_i\|_p : i \in \mathbb{N}_n\}$.

Proof. Set $X_i = \operatorname{supp} f_i \ (i \in \mathbb{N}_n)$, and set $C = \max\{\|f_i\|_q \|x_i\|_p : i \in \mathbb{N}_n\}$. For $a \in L^1(\Omega)$, we have

$$\begin{aligned} \|T(a)\|_p^p = \left\| \sum_{i=1}^{n} \langle U(a), f_i \rangle \, x_i \right\|_p^p &= \sum_{i=1}^{n} |\langle U(a), f_i \rangle|^p \, \|x_i\|_p^p \\ &\leq \sum_{i=1}^{n} \|\chi_{X_i} U(a)\|_p^p \, \|f_i\|_q^p \, \|x_i\|_p^p \\ &\leq C^p \, \|\chi_{X_1 \cup \cdots \cup X_n} U(a)\|_p^p \leq C^p \, \|U(a)\|_p^p . \end{aligned}$$

Therefore $\|T(a)\|_p \leq C \, \|U(a)\|_p$, and the result follows. □

Lemma 5.2.6. *Let G be a locally compact group, and let $s_1, \dots, s_n \in G$. Then there exists an open, relatively compact neighbourhood V of e_G such that the sets $s_1V, \dots, s_nV$ are pairwise disjoint.*

Proof. Since G is a Hausdorff space there exist pairwise disjoint open sets $U_1, \dots, U_n$ with $s_i \in U_i$ $(i \in \mathbb{N}_n)$. For each $i \in \mathbb{N}_n$ the map $t \mapsto s_it$ is continuous at e_G and so there exists an open neighbourhood V_i of e_G with $s_iV_i \subset U_i$ $(i \in \mathbb{N}_n)$. Set $V = \cap V_i$. Then V is the required set. □

In the theorem below we shall use the following identity. For each $x \in L^p(G)$, $\lambda \in L^\infty(G)$ and $s \in G$ we have

$$x \otimes (\lambda \cdot s) = s^{-1} * [(s \cdot x) \otimes \lambda] . \tag{5.6}$$

Theorem 5.2.7. *Let G be a locally compact group, and take p with $1 < p < \infty$. Suppose that $L^p(G)$ is injective in $L^1(G)$-**mod**. Then G is super p-amenable.*

Proof. We set $A = L^1(G)$, $E = L^p(G)$ and $J = \mathcal{B}(A, E)$. By Proposition 5.2.2 there exists $R \in {}_A\mathcal{B}(\widetilde{J}, E)$ with $R \circ \widetilde{\Pi} = I_E$. Set $C = \|R\|$. For a compact, non-null set $V \subset G$ we define a linear functional Λ_V on A' by

$$\langle \lambda, \Lambda_V \rangle = \frac{1}{m(V)} \int_V R(\chi_V \otimes \lambda)(t) \, \mathrm{d}m(t) \quad (\lambda \in L^\infty(G)) .$$

For $\lambda \in A'$ we have

$$|\langle \lambda, \Lambda_V \rangle| \le \|R(\chi_V \otimes \lambda)\|_p \|\chi_V / m(V)\|_q \le C \|\lambda\|_\infty \|\chi_V\|_p \|\chi_V / m(V)\|_q = C \|\lambda\|_\infty ,$$

and so $\Lambda_V \in A''$ with $\|\Lambda_V\| \le C$. Let $\mathcal{F}$ be the family of compact, non-null neighbourhoods of e_G in G, and set $V_1 \le V_2$ if $V_2 \subset V_1$. Then $(\mathcal{F}, \le)$ is a directed set. Let Λ be a weak-$*$ accumulation point in A'' of the bounded net $\{\Lambda_V : V \in \mathcal{F}\}$. We *claim* that Λ is super p-invariant.

Clearly $\langle 1, \Lambda \rangle = 1$ since for each $V \in \mathcal{F}$ we have

$$\langle 1, \Lambda_V \rangle = \frac{1}{m(V)} \int_V R(\widetilde{\Pi}\chi_V)(t) \, \mathrm{d}m(t) = \frac{1}{m(V)} \int_V \mathrm{d}m(t) = 1 .$$

Take $n \in \mathbb{N}$ and a finite subset $\{s_1, \dots, s_n\} \subset G$. Let $U \in \mathcal{B}(A, E)$, and let $\mathbf{X} = (X_1, \dots, X_n)$ be a measurable partition of G. Take $f_1, \dots, f_n \in L^q(G)_{[1]}$ with $\operatorname{supp} f_i \subset X_i$ $(i \in \mathbb{N}_n)$. Choose $V \in \mathcal{F}$ such that the sets $s_1V, \dots, s_nV$ are pairwise disjoint. Set

$$T = \sum_{i=1}^{n} \chi_{s_iV} \otimes U'(f_i) \in \mathcal{L}(A, E) .$$

By Lemma 5.2.5, $T \in J$ and $\|T\| \le \|U\| \, m(V)^{1/p}$.

For each $i \in \mathbb{N}_n$ we have

$$\begin{aligned} m(V)\left\langle f_i, U''(s_i \cdot \Lambda_V)\right\rangle &= m(V)\left\langle U'(f_i), s_i \cdot \Lambda_V\right\rangle \\ &= \int_V R(\chi_V \otimes (U'(f_i)\cdot s_i))(t)\,\mathrm{d}m(t) \\ &= \int_V R((s_i\cdot\chi_V)\otimes U'(f_i))(s_i t)\,\mathrm{d}m(t) \quad \text{(by (5.6))} \\ &= \int_{s_iV} R(\chi_{s_iV}\otimes U'(f_i))(t)\,\mathrm{d}m(t) \\ &= \int_{s_iV} R(\chi_{s_iV}T)(t)\,\mathrm{d}m(t)\,. \end{aligned}$$

Hence by Hölder 's inequality we have

$$|\langle U'(f_i), s_i\cdot\Lambda_V\rangle| \le \|\chi_{s_iV}R(\chi_{s_iV}T)\|_p\, m(V)^{1/q-1}\,.$$

Then by Proposition 5.2.4 we have

$$\begin{aligned} \left(\sum_{i=1}^n |\langle f_i, U''(s_i\cdot\Lambda_V)\rangle|^p\right)^{1/p} &\le \left(\sum_{i=1}^n \|\chi_{s_iV}R(\chi_{s_iV}T)\|_p^p\right)^{1/p} m(V)^{1/q-1} \\ &\le C\,\|T\|\,m(V)^{1/q-1} \le C\,\|U\|\,. \end{aligned}$$

Therefore

$$\left(\sum_{i=1}^n |\langle f_i, U''(s_i\cdot\Lambda)\rangle|^p\right)^{1/p} = \lim_V \left(\sum_{i=1}^n |\langle U'(f_i), s_i\cdot\Lambda_V\rangle|^p\right)^{1/p} \le C\,.$$

Since this is true for all such collections (f_i), we have

$$\left(\sum_{i=1}^n \|\chi_{X_i}U''(s_i\cdot\Lambda)\|_p^p\right)^{1/p} \le C\,.$$

Since this is true for each measurable partition $\mathbf{X}$ and $U \in J_{[1]}$, by Lemma 4.6.20, we have

$$\|(s_1\cdot\Lambda,\ldots,s_n\cdot\Lambda)\|_n^{\mathcal{B}} \le C\,.$$

Therefore G is super p-amenable. □

5.2.3 The discrete case

The proof of Theorem 5.2.7 becomes much simpler when G is discrete. Let G be a group, and let $1 < p < \infty$. We set $A = \ell^1(G)$, $E = \ell^p(G)$ and $J = \mathcal{B}(A, E)$. We shall identify J with a space of functions in $\mathbb{C}^{G\times G}$ via

$$U(t,s) = U(\delta_t)(s) \quad (s,t \in G,\, U \in J)\,.$$

With this identification we have

$$(r * U)(s,t) = U(r^{-1}s, r^{-1}t) \quad (r,s,t \in G,\, U \in J)\,.$$

The following is a special case of Proposition 5.2.2, but the proof becomes much more direct when G is discrete.

Proposition 5.2.8. *Let G be a group, and let $1 < p < \infty$. Then $\ell^p(G)$ is injective in $\ell^1(G)$-**mod** if and only if the morphism $\widetilde{\Pi} : E \to \widetilde{J}$ is a coretraction.*

Proof. The condition is necessary by [19, VII.1.34]. We shall show it is sufficient.

Suppose that there is a morphism $R \in {}_A\mathcal{B}(\widetilde{J}, E)$ with $R \circ \widetilde{\Pi} = I_E$. Define $Q : J \to \widetilde{J}$ by

$$Q(U)(a) = (a * U)(\delta_e) \quad (a \in A,\, U \in J)\,.$$

For $U \in J$ we have

$$Q(U)(t,s) = (t * U)(e,s) = U(t^{-1}, t^{-1}s) \quad (t, s \in G)\,.$$

Now for $r \in G$ we have

$$\begin{aligned}(r * Q(U))\,(t,s) &= Q(U)(r^{-1}t, r^{-1}s) = U(t^{-1}r, t^{-1}s)\\ &= (r \cdot U)\,(t^{-1}, t^{-1}s) = Q(r \cdot U)(t,s)\,,\end{aligned}$$

and hence Q is a left A-module morphism. For $x \in E$, we have

$$Q(\Pi(x))(t,s) = \Pi(x)(t^{-1}, t^{-1}s) = (t^{-1} \cdot x)(t^{-1}s) = x(s) = \widetilde{\Pi}(x)(t,s) \quad (s, t \in G)\,.$$

Hence $Q \circ \Pi = \widetilde{\Pi}$. Finally, we set $\rho = R \circ Q$. Then $\rho \in {}_A\mathcal{B}(J, E)$ and $\rho \circ \Pi = I_E$, and so E is injective in A-**mod**. □

Proposition 5.2.9. *Let S be a set, and take $1 < p < \infty$. Let $R : \mathcal{B}(\ell^1(S), \ell^p(S)) \to \ell^p(S)$ be a bounded linear operator. Then*

$$\left(\sum_{s \in S} |R(\delta_s U)(s)|^p\right)^{1/p} \leq \|U\|\,\|R\| \quad (U \in \mathcal{B}(\ell^1(S), \ell^p(S)))\,.$$

Proof. This follows from Proposition 5.2.4. □

By equation (5.6) we have

$$\delta_e \otimes (U'(\delta_s) \cdot s) = s^{-1} * [\delta_s \otimes U'(\delta_s)] = s^{-1} * (\delta_s U)\,. \tag{5.7}$$

Theorem 5.2.10. *Let G be a group, and take p with $1 < p < \infty$. Then $\ell^p(G)$ is injective in $\ell^1(G)$-**mod** if and only if G is super p-amenable.*

Proof. We set $A = \ell^1(G)$ and $E = \ell^p(G)$. Suppose first that E is injective in A-**mod**. By Proposition 5.2.8 there exists $R \in {}_A\mathcal{B}(\widetilde{J}, E)$ with $R \circ \widetilde{\Pi} = I_E$. We set $C = \|R\|$ and define $\Lambda \in A''$ by

$$\langle \lambda, \Lambda \rangle = R(\delta_e \otimes \lambda)(e) \quad (\lambda \in A')\,.$$

We have

$$\langle 1, \Lambda \rangle = R(\delta_e \otimes \chi_G)(e) = R(\widetilde{\Pi}\delta_e)(e) = \delta_e(e) = 1\,.$$

We *claim* that Λ is a super p-invariant mean. Take $n \in \mathbb{N}$ and a finite subset $\{s_1, \ldots, s_n\} \subset G$. Let $U \in \mathcal{B}(A, E)$, and let $\mathbf{X} = (X_1, \ldots, X_n)$ be a partition of G. Take $f_1, \ldots, f_n \in E'_{[1]}$ with supp $f_i \subset X_i$ $(i \in \mathbb{N}_n)$. Set

$$T = \sum_{i=1}^{n} \delta_{s_i} \otimes U'(f_i) \in \mathcal{L}(A, E)\,.$$

By Lemma 5.2.5 $T \in J$ and $\|T\| \leq \|U\|$.

For each $i \in \mathbb{N}_n$ we have

$$\begin{aligned}\langle f_i, U''(s_i \cdot \Lambda)\rangle &= \langle U'(f_i), s_i \cdot \Lambda \rangle = R(\delta_e \otimes (U'(f_i) \cdot s_i))(e) \\ &= R(\delta_{s_i} \otimes U'(f_i))(s_i) \quad \text{(by (5.7))} \\ &= R(\delta_{s_i} T)(s_i)\,.\end{aligned}$$

Then by Proposition 5.2.4 we have

$$\left(\sum_{i=1}^{n} |\langle f_i, U''(s_i \cdot \Lambda)\rangle|^p\right)^{1/p} = \left(\sum_{i=1}^{n} |R(\delta_{s_i} T)(s_i)|^p\right)^{1/p} \leq C\,\|U\|$$

Since this is true for all such collections (f_i), we have

$$\left(\sum_{i=1}^{n} \|\chi_{X_i} U''(s_i \cdot \Lambda)\|_p^p\right)^{1/p} \leq C\,\|U\|\ .$$

Since this is true for each measurable partition $\mathbf{X}$ and $U \in J$, by Lemma 4.6.20 we have

$$\|(s_1 \cdot \Lambda, \ldots, s_n \cdot \Lambda)\|_n^{\mathcal{B}} \leq C\,.$$

Therefore G is super p-amenable.

Conversely, suppose that G is super p-amenable, and let $\Lambda \in A''$ be a super p-invariant mean. For $U \in J$ define $R(U) : G \to \mathbb{C}$ by

$$R(U)(s) = \langle U'(\delta_s), s \cdot \Lambda \rangle = U''(s \cdot \Lambda)(s) \quad (s \in G)\,.$$

It is easily checked that $R(U) \in E$ and $R \in \mathcal{B}(J, E)$ with $\|R\| \leq C$. For $r, s \in G$ we have

$$R(r * U)(s) = U''(r^{-1}s \cdot \Lambda)(r^{-1}s) = [r \cdot R(U)]\,(s)\,.$$

Therefore $R \in {}_A\mathcal{B}(J, E)$. For $x \in E$ and $s \in G$ we have

$$R(\widetilde{\Pi}x)(s) = \langle 1, \Lambda \rangle\, x(s) = x(s)\,.$$

Therefore $R \circ \widetilde{\Pi} = I_E$. By Proposition 5.2.8, E is injective in A-**mod**. □

Chapter 6

Biflatness of $\ell^1(S)$

Now we turn our attention to discrete semigroup algebras. In this chapter we prove results about the projectivity and flatness of $\ell^1(S)$ for a semigroup S. The results about biflatness are taken from the article [30].

In [2] the biflatness of the Banach algebra $\ell^1(S)$ was characterized for a Clifford semigroup S. Indeed the following theorem was proved.

Theorem 6.0.11 ([2, Theorem 6.1])**.** *Let S be a Clifford semigroup. Then $\ell^1(S)$ is biflat if and only if:*

(i) *$(E(S), \leq)$ is uniformly locally finite; and*

(ii) *each maximal subgroup is amenable.* □

The proof of this theorem uses the representation theory for inverse semigroups developed in [36] and [37], but adapted to the Banach algebra setting.

We shall prove in Theorem 6.2.4 that, for any semigroup S such that $\ell^1(S)$ is biflat, $(E(S), \leq)$ is uniformly locally finite. This is similar to a theorem of Duncan and Paterson [12, Theorem 2] ($\ell^1(S)$ amenable $\Rightarrow E(S)$ finite). This theorem allows us to extend the method of [2] to characterize the biflatness and biprojectivity of $\ell^1(S)$ for the class of inverse semigroups (Theorem 6.2.7).

A biflat Banach algebra is weakly amenable [4, Proposition 2.8.62]. Hence our results provide further examples of non-commutative Banach algebras that are weakly amenable.

6.1 The Banach algebra $\ell^1(S)$ for a ULF inverse semigroup

We shall show that for a uniformly locally finite (ULF) inverse semigroup the Banach algebra $\ell^1(S)$ is isomorphic to a direct sum of ℓ^1-Munn algebras over group algebras. This an adaption to the Banach algebra setting of [37, Theorem 4.6] using [2, Proposition 6.5]. Since the terminology of [37] is different from ours and to aid the reader we give some of the details.

Let S be an inverse semigroup. We define a multiplication $*$ on the space $\ell^1(S)$ by

$$\delta_s * \delta_t = \begin{cases} \delta_{st}, & \text{if } s^{-1}s = tt^{-1} \\ 0, & \text{otherwise} \end{cases} .$$

A direct check shows that this defines an associative product. We denote the resulting Banach algebra by $B(S)$. Let D be a $\mathcal{D}$-class of S. We set $B(D) = \ell^1(D)$, regarded as a subalgebra of $B(S)$.

Proposition 6.1.1. *Let S be an inverse semigroup.*

(i) *$B(D)$ is an ideal in $B(S)$ for each $\mathcal{D}$-class D.*

(ii) *Let $\{D_\lambda : \lambda \in \Lambda\}$ be the family of $\mathcal{D}$-classes of S indexed by some set Λ. There is an isomorphism of Banach algebras*

$$B(S) \cong \ell^1\text{-}\bigoplus\{B(D_\lambda) : \lambda \in \Lambda\}\,.$$

Proof. (i) Fix a $\mathcal{D}$-class D. Take $s \in S$ and $t \in D$. If $\delta_s * \delta_t = 0$, then $\delta_s * \delta_t \in B(D)$. Otherwise $\delta_s * \delta_t = \delta_{st}$ and $s^{-1}s = tt^{-1}$. Then we have

$$(st)^{-1}(st) = t^{-1}s^{-1}st = t^{-1}tt^{-1}t = t^{-1}t\,.$$

Setting $x = t^{-1}t$ in Proposition 1.6.10 we see that $st\,\mathcal{D}\,t$. Hence $\delta_s * \delta_t \in B(D)$ and $B(D)$ is a left ideal. Similarly it is a right ideal.

(ii) Clearly there is an isomorphism of Banach spaces. We just need to show that, if s and t lie in different $\mathcal{D}$-classes, then $\delta_s * \delta_t = 0$. Suppose that $\delta_s * \delta_t \neq 0$. Then $s^{-1}s = tt^{-1}$ and so $s\,\mathcal{D}\,t^{-1}$. Since $t^{-1}\,\mathcal{D}\,t$ we have $s\,\mathcal{D}\,t$. □

Theorem 6.1.2 (*cf.* [37, Theorem 4.5])**.** *Let S be a uniformly locally finite inverse semigroup with $\mathcal{D}$-classes $\{D_\lambda : \lambda \in \Lambda\}$. Then there is an isomorphism of Banach algebras*

$$\ell^1(S) \cong \ell^1\text{-}\bigoplus\{B(D_\lambda) : \lambda \in \Lambda\}\,.$$

Proof. Let $B(S)$ be the Banach algebra defined above. By Proposition 1.6.4 the Schützenberger representation Sch : $\ell^1(S) \to B(S)$ is an isomorphism of Banach spaces. It is proved in [37, Lemma 4.1] that Sch is an algebra homomorphism. The result now follows from Proposition 6.1.1. □

We now turn to identifying the Banach algebra $B(D)$ for a $\mathcal{D}$-class D.

Theorem 6.1.3. *Let S be an inverse semigroup, and let $D \subset S$ be a $\mathcal{D}$-class. Take $\widetilde{p} \in E(D)$. Then there is an isometric isomorphism of Banach algebras*

$$B(D) \cong \mathbb{M}_{E(D)}(\ell^1(G_{\widetilde{p}}))\,.$$

Proof. Set $G = G_{\widetilde{p}}$ which, up to isomorphism, does not depend on the choice of $\widetilde{p}$. For $p, q \in E(D)$ define

$$X_{p,q} = \{s \in D : ss^{-1} = p,\ s^{-1}s = q\}\,.$$

We have $X_{p,p} = G_p$. For each $p \in E(D)$ we use Proposition 1.6.10 to choose $x_p \in S$ with $\widetilde{p} = x_p x_p^{-1}$ and $p = x_p^{-1} x_p$. Then the map

$$\theta_{p,q} : s \mapsto x_p s x_q^{-1}, \quad X_{p,q} \to G\,,$$

extends to an isometric isomorphism of Banach spaces $\theta_{p,q} : \ell^1(X_{p,q}) \to \ell^1(G)$. Let $P_{p,q} : B(D) \to \ell^1(X_{p,q})$ be the natural projection. Define

$$\theta : B(D) \to \mathbb{M}_{E(D)}(\ell^1(G))$$

by

$$\theta(a) = \sum_{p,q \in E(D)} \theta_{p,q} \circ P_{p,q}(a) E_{p,q} \quad (a \in B(D))\,.$$

Then θ is an isometric isomorphism of Banach spaces. For $p, q, u, v \in E(D)$ we have

$$\ell^1(X_{p,q}) * \ell^1(X_{u,v}) = 0 \text{ if } q \neq u, \text{ and } \ell^1(X_{p,q}) * \ell^1(X_{q,v}) = \ell^1(X_{p,v})\,.$$

It follows from this that θ is a Banach algebra isomorphism. □

Theorem 6.1.4 (*cf.* [37, Theorem 4.6])**.** *Let S be a uniformly locally finite inverse semigroup with $\mathcal{D}$-classes $\{D_\lambda : \lambda \in \Lambda\}$. For each λ take an idempotent $p_\lambda \in D_\lambda$. Then there is an isomorphism of Banach algebras*

$$\ell^1(S) \cong \ell^1\text{-}\bigoplus\{\mathbb{M}_{E(D_\lambda)}(\ell^1(G_{p_\lambda})) : \lambda \in \Lambda\}\,.$$

Proof. This follows by combining Theorem 6.1.2 and Theorem 6.1.3. □

Suppose that S is a Clifford semigroup. Then each $\mathcal{D}$-class D contains a single idempotent p (say) and $B(D) = \ell^1(G_p)$.

Let S be an inverse semigroup. Then the natural involution $s \mapsto s^{-1}$ lifts to an isometric involution on $\ell^1(S)$. Hence $\ell^1(S)$ has a bounded left approximate identity if and only if $\ell^1(S)$ has a bounded right approximate identity. The following proposition shows that locally finite properties of S are not useful for studying $\ell^1(S)$ when $\ell^1(S)$ has an identity.

Proposition 6.1.5. *Let S be an inverse semigroup. Suppose that either:*

(i) *S is locally finite, and $\ell^1(S)$ has an identity, or*

(ii) *S is uniformly locally finite, and $\ell^1(S)$ has a bounded approximate identity.*

Then $E(S)$ is finite.

Proof. (i) By [11, Lemma 13] $\ell^1(S)$ has an identity if and only if $\ell^1(E(S))$ has an identity. Therefore by Proposition 1.6.8 it is sufficient to prove the result in the case where S is a semilattice, and so we suppose that this is the case.

Let $e_A = \sum_{s\in S} e_s\delta_s$ be an identity for $\ell^1(S)$. For each $t \in S$, the equation $\delta_t \star e_A = \delta_t$ implies that

$$\sum_{s\in[t)} e_s = 1\,.$$

Let M be the set of maximal elements of S. For each $m \in M$ we have $[m) = \{m\}$. It follows that there are only finitely many maximal elements.

There is a finite set $F \subset S$ such that

$$\sum_{s\in S\setminus F} |e_s| < 1/2\,.$$

Take $t \in S$ and assume towards a contradiction that there exists an infinite increasing chain

$$t < s_1 < s_2 < \cdots$$

in S. Let $f \in F$. Since S is locally finite, the inequality $s_n \leq f$ holds for only finitely many $n \in \mathbb{N}$. Set

$$N = \max_{f\in F} \{\max\{n \in \mathbb{N} : f \in [s_n)\}\}\,.$$

Then $F \cap [s_{N+1}) = \emptyset$. However $\sum_{s\in[s_{N+1})} e_s = 1$, which is a contradiction. Therefore, for each $t \in S$, there exists $m \in M$ with $t \leq m$. Hence $S \subset \bigcup_{m\in M}(m]$ and S is finite.

(ii) This is similar to part (i). Again by [11, Lemma 13] we may suppose that S is a semilattice, and an easy modification of the above argument shows that there are finitely many maximal elements. From the hypothesis that S is uniformly locally finite, $S \subset \bigcup_{m\in M}(m]$. Therefore S is finite. □

Example 6.1.6. The semigroup $\mathbb{N}_\wedge$ is locally finite but not uniformly locally finite. The sequence $(\delta_n)_{n\in\mathbb{N}}$ is a bounded approximate identity for $\ell^1(\mathbb{N}_\wedge)$, but $\ell^1(\mathbb{N}_\wedge)$ does not have an identity.

6.2 Main results

Let S be a semigroup, and set $A = \ell^1(S)$. Suppose that A is biflat. Then there is an A-bimodule morphism $\rho : A \to (A \widehat{\otimes} A)''$ with $\pi'' \circ \rho = i_A$. Fix $u \in S$. Suppose that $ru = vw$ for some elements $r, v, w \in S$, and set $\theta = ru = vw$. We can find nets $(\widetilde{z_\alpha})$ and $(\widetilde{w_\alpha})$ in $(A \widehat{\otimes} A)_{[\|\rho\|]}$ indexed by the same directed set such that $\lim_\alpha \widetilde{z_\alpha} = \rho(\delta_u)$ and $\lim_\alpha \widetilde{w_\alpha} = \rho(\delta_v)$ in the weak-$*$ topology. In particular we have

$$1 = \langle \Pi(\lambda_\theta), \rho(\delta_\theta)\rangle = \lim_\alpha \langle \Pi(\lambda_\theta), \delta_r \cdot \widetilde{z_\alpha}\rangle = \lim_\alpha \langle \lambda_\theta, \pi(\delta_r \cdot \widetilde{z_\alpha})\rangle \ .$$

The bounded net

$$(\delta_r \cdot \widetilde{z_\alpha} - \widetilde{w_\alpha} \cdot \delta_w, \langle \Pi(\lambda_\theta), \delta_r \cdot \widetilde{z_\alpha}\rangle - 1)$$

converges to 0 in the space $(A \widehat{\otimes} A) \oplus_\infty \mathbb{C}$ with respect to the weak topology. By Mazur's theorem there exists a net $u_\alpha \subset \langle (\delta_r \cdot \widetilde{z_\alpha} - \widetilde{w_\alpha} \cdot \delta_w, \langle \Pi(\lambda_\theta), \delta_r \cdot \widetilde{z_\alpha}\rangle - 1)\rangle$ such that $u_\alpha \to 0$ in norm in the space $(A \widehat{\otimes} A) \oplus_\infty \mathbb{C}$. For each index α we can write

$$\begin{aligned} u_\alpha &= \sum_{i=1}^n t_i^\alpha (\delta_r \cdot \widetilde{z_{\alpha_i}} - \widetilde{w_{\alpha_i}} \cdot \delta_w, \langle \Pi(\lambda_\theta), \delta_r \cdot \widetilde{z_{\alpha_i}}\rangle - 1) \\ &= \left(\delta_r \cdot \left(\sum_{i=1}^n t_i^\alpha \widetilde{z_{\alpha_i}}\right) - \left(\sum_{i=1}^n t_i^\alpha \widetilde{w_{\alpha_i}}\right) \cdot \delta_w, \left\langle \Pi(\lambda_\theta), \delta_r \cdot \sum_{i=1}^n t_i^\alpha \widetilde{z_{\alpha_i}} \right\rangle - 1\right) . \end{aligned}$$

with $\sum_{i=1}^n t_i^\alpha = 1$. Set $z_\alpha = \sum_{i=1}^n t_i^\alpha \widetilde{z_{\alpha_i}}$ and $w_\alpha = \sum_{i=1}^n t_i^\alpha \widetilde{w_{\alpha_i}}$. Then the nets (z_α) and (w_α) satisfy

$$\lim_\alpha \|\delta_r \cdot z_\alpha - w_\alpha \cdot \delta_w\|_\pi = 0 \,, \tag{6.1}$$

$$\langle \Pi(\lambda_\theta), \delta_r \cdot z_\alpha\rangle \to 1 \,. \tag{6.2}$$

Using the identification $A \widehat{\otimes} A = \ell^1(S \times S)$, for each α we can write $z_\alpha = \sum_{s,t \in S} z_{s,t}^\alpha \delta_{(s,t)}$ and $w_\alpha = \sum_{s,t\in S} w_{s,t}^\alpha \delta_{(s,t)}$ where $(z_{s,t}^\alpha), (w_{s,t}^\alpha) \subset \mathbb{C}$.

Lemma 6.2.1. *We have*

$$\lim_\alpha \sum_{(y,t)\in Z(r,w,\theta)} z_{y,t}^\alpha = 1 \,, \tag{6.3}$$

where $Z(r, w, \theta) = \{(y, t) \in S \times S : t \in Sw, ryt = \theta\}$.

Proof. From equation (6.1) we have

$$0 = \lim_\alpha (\delta_r \cdot z_\alpha - w_\alpha \cdot \delta_w) = \lim_\alpha \sum_{s,t\in S} \left(\sum_{y\in[r^{-1}s]} z_{y,t}^\alpha - \sum_{x \in [tw^{-1}]} w_{s,x}^\alpha \right) \delta_{(s,t)} \,. \tag{6.4}$$

Since $S \setminus Sw = \{t \in S : [tw^{-1}] = \emptyset\}$, taking the norm of this expression and removing the summands for $t \in Sw$ gives

$$\lim_{\alpha} \sum_{s\in S} \sum_{t\in S\setminus Sw} \left| \sum_{y\in[r^{-1}s]} z^{\alpha}_{y,t} \right| = 0\,.$$

Removing the summands for $t \in S \setminus [s^{-1}\theta]$, and the modulus sign gives

$$\lim_{\alpha} \sum_{s\in S} \sum_{t\in(S\setminus Sw)\cap[s^{-1}\theta]} \sum_{y\in[r^{-1}s]} z^{\alpha}_{y,t} = 0\,.$$

It is easily checked that

$$\left\{(y,t) \in S \times S : \exists s \in S,\, t \in [s^{-1}\theta],\, y \in [r^{-1}s]\right\} = \{(y,t) \in S \times S : ryt = \theta\}\ .$$

Hence

$$\lim_{\alpha} \sum_{\{(y,t)\in S\times S:\, t\in S\setminus Sw,\, ryt=\theta\}} z^{\alpha}_{y,t} = 0\,. \tag{6.5}$$

From equation (6.2) we have

$$\lim_{\alpha} \sum_{\{(y,t)\in S\times S: ryt=\theta\}} z^{\alpha}_{y,t} = 1\,. \tag{6.6}$$

Now combining equations (6.5) and (6.6) gives

$$\lim_{\alpha} \sum_{(y,t)\in Z(r,w,\theta)} z^{\alpha}_{y,t} = 1\,, \tag{6.7}$$

where $Z(r,w,\theta) = \{(y,t) \in S \times S : t \in Sw,\, ryt = \theta\}$. In particular this latter set is non-empty. □

Theorem 6.2.2. *Let S be a semigroup. Suppose that the Banach algebra $\ell^1(S)$ is biflat. Then there is a constant $C > 0$ such that the following property holds: for each $u \in S$, $N \in \mathbb{N}$ and elements $(r_1, v_1, w_1), \ldots, (r_N, v_N, w_N) \in S \times S \times S$ such that:*

(i) $r_i u = v_i w_i \ (i = 1, \ldots, N)$*; and,*

(ii) *the sets* $Sw_1 \cap [r_1^{-1}(r_1u)], \ldots, Sw_N \cap [r_N^{-1}(r_Nu)]$ *are pairwise disjoint.*

Then necessarily $N \le C$.

Proof. Set $A = \ell^1(S)$. Since A is biflat, there exists an A-bimodule morphism $\rho : A \to (A \widehat{\otimes} A)''$ with $\pi \circ \rho = i_A$. Set $C = \|\rho\|$. For each $i \in \mathbb{N}_N$ we set $\theta_i = r_i u = v_i w_i$. Consider the set $Z(r_i, w_i, \theta_i)$ as defined in Lemma 6.2.1. If $(y,t) \in Z(w_i, u_i, \theta_i)$, then $yt \in Sw_i \cap [r_i^{-1}\theta_i]$. It follows that the sets $Z(w_i, u_i, \theta_i)$ are pairwise disjoint. Now summing equation (6.3) over the sets $Z_i = Z(r_i, w_i, \theta_i)$ gives

$$N = \lim_{\alpha} \sum_{i=1}^{N} \sum_{(y,t)\in Z_i} z^{\alpha}_{y,t} \le \sup_{\alpha} \sum_{i=1}^{N} \sum_{(y,t)\in Z_i} \left|z^{\alpha}_{y,t}\right| \le \|\rho(\delta_u)\|_{\pi} = \|\rho\| = C\,.$$

The result follows. □

Remark 6.2.3. The same theorem holds (and with the same constant) if the sets $Sw_i \cap [r_i^{-1}(r_i u)]$ are replaced by $r_i S \cap [(r_i u)w_i^{-1}]$. We obtain this by restricting the sum in equation (6.4) to $s \in (S \setminus Sr) \cap [\theta t^{-1}]$ and then following through the same argument.

We now define two relations on the set $E(S)$. We set $u\, R\, v$ if $u \in Sv \cap [v^{-1}v]$ and $u\, \widetilde{R}\, v$ if $u \in vS \cap [vv^{-1}]$. Clearly $u\, R\, u$ and $u\, \widetilde{R}\, u$. Suppose that $u\, R\, v$. Then $u = sv$ for some $s \in S$ and $vu = v$. Hence $v \in Su$ and $uv = svv = sv = u$. We see that $v\, R\, u$, and so R is symmetric. Similarly $\widetilde{R}$ is symmetric. A straight forward check shows that R and $\widetilde{R}$ are transitive. Thus R and $\widetilde{R}$ are equivalence relations on $E(S)$. The R-class containing u is $Su \cap [u^{-1}u]$ and the $\widetilde{R}$-class containing u is $uS \cap [uu^{-1}]$.

Theorem 6.2.4. *Let S be a semigroup such that $\ell^1(S)$ is biflat. Then $E(S)$ is uniformly locally finite.*

Proof. Let C be the constant given in Theorem 6.2.2. Fix $u \in E(S)$. The collection $\mathcal{A} = \{St \cap [t^{-1}t] : t \in (u]\}$ is a partition of $(u]$. By Theorem 6.2.2, $|\mathcal{A}| \leq C$. Each R-class is a left-zero semigroup and each $\widetilde{R}$-class is a right-zero semigroup. It follows that if $x \neq y$ lie in the same R-class then x and y lie in different $\widetilde{R}$-classes. Assume that one of the classes in $\mathcal{A}$ contains $C + 1$ or more elements. Then we obtain a partition of $(u]$ by at least $C + 1$ $\widetilde{R}$-classes. This contradicts (the remark following) Theorem 6.2.2. Hence each class in $\mathcal{A}$ contains at most C elements and $|(u]| < C^2$. This holds for each $u \in E(S)$ and so $E(S)$ is uniformly locally finite. □

Remark 6.2.5. The above theorem shows that if $\ell^1(S)$ is C-biflat, then S is locally C^2-finite. We can improve this estimate for semigroups with commuting idempotents. Indeed, let S be a semigroup such that the idempotents commute. Then $Su \cap [u^{-1}u] = uS \cap [uu^{-1}] = \{u\}$ $(u \in E(S))$. Hence the argument above shows that if $\ell^1(S)$ is C-biflat, then S is locally C-finite. This agrees with the result in [2] for Clifford semigroups.

A similar argument gives a quantitative version of [12, Theorem 2]. We sketch the details.

Theorem 6.2.6. *Let S be a semigroup such that $\ell^1(S)$ is amenable. Then*

$$|E(S)|^{1/2} \leq AM(\ell^1(S)) .$$

If the idempotents of S commute, then

$$|E(S)| \leq AM(\ell^1(S)) .$$

Proof. Let $\mathcal{A}$ and $\widetilde{\mathcal{A}}$ denote the sets of R-classes and $\widetilde{R}$-classes respectively. The proof of [12, Theorem 2] shows that $|\mathcal{A}| \leq AM(\ell^1(S))$ and $\left|\widetilde{\mathcal{A}}\right| \leq AM(\ell^1(S))$. Now the result follows using the same arguments as above. □

A better estimate is known for Clifford semigroups; it follows from [15, Theorem 2.2 and Corollary 1.8] that

$$2\,|E(S)| - 1 \leq AM(\ell^1(E(S))) \leq AM(\ell^1(S))\,.$$

The following theorem is an extension of [2, Theorem 6.1]. The proof in [2] that for a Clifford semigroup S, $\ell^1(S)$ biflat implies S is uniformly locally finite uses the fact that $\ell^1(E(S))$ is a retract of $\ell^1(S)$. This is not true for a general inverse semigroup, and so we have to use Theorem 6.2.4.

Theorem 6.2.7. *Let S be an inverse semigroup. Then:*

(i) *$\ell^1(S)$ is biflat if and only if S is uniformly locally finite and G_p is amenable for each $p \in E(S)$;*

(ii) *$\ell^1(S)$ is biprojective if and only if S is uniformly locally finite and G_p is finite for each $p \in E(S)$.*

Proof. (i) Suppose that $\ell^1(S)$ is biflat. Then, by Theorem 6.2.4 and Proposition 1.6.8, S is uniformly locally finite. By Theorem 6.1.4 there is an isomorphism of Banach algebras

$$\ell^1(S) \cong \ell^1\text{-}\bigoplus\{\mathbb{M}_{E(D_\lambda)}(\ell^1(G_{p_\lambda})) : \lambda \in \Lambda\}\,,$$

where $\{D_\lambda : \lambda \in \Lambda\}$ are the $\mathcal{D}$-classes of S and $p_\lambda \in E(D_\lambda)$ $(\lambda \in \Lambda)$. By Proposition 2.3.7 and Proposition 2.3.11 each Banach algebra $\ell^1(G_{p_\lambda})$ $(\lambda \in \Lambda)$ is biflat. Hence by Johnson's theorem each group G_{p_λ} $(\lambda \in \Lambda)$ is amenable. Now for any $p \in E(S)$ there exists λ such that $p\,\mathcal{D}\,p_\lambda$. Then $G_p \cong G_{p_\lambda}$ and hence G_p is amenable.

Conversely, suppose that the conditions on S hold. By Johnson's theorem each Banach algebra $\ell^1(G_{p_\lambda})$ $(\lambda \in \Lambda)$ is 1-biflat. By Proposition 2.3.11, each Banach algebra $\mathbb{M}_{E(D_\lambda)}(\ell^1(G_{p_\lambda}))$ $(\lambda \in \Lambda)$ is 1-biflat. By Proposition 2.3.7 and Theorem 6.1.4 $\ell^1(S)$, is biflat.

(ii) This follows in the same way except we use the fact that $\ell^1(G)$ is biprojective if and only if G is finite, in which case $\ell^1(G)$ is 1-biprojective [4, 3.3.32]. □

6.2.1 Regularity of S

In [1] the weak amenability of $\ell^1(S)$ is studied and some necessary conditions for weak amenability are obtained. These are conditions concerning the number of singular elements of S. Since biflatness implies weak amenability the results in [1] apply directly to semigroups such that $\ell^1(S)$ is biflat. However, we can use the stronger property of biflatness to obtain stronger conditions with simpler proofs.

Lemma 6.2.8. *Let S be a semigroup such that $\ell^1(S)$ is biflat. Let $t \in S$ with $[tt^{-1}] \neq \emptyset$ and $[t^{-1}t] \neq \emptyset$. Then t is regular.*

Proof. There are elements $u, v \in S$ with $tu = vt = t$. Lemma 6.2.1 shows that the set $Z(t,t,t) \neq \emptyset$, and hence $St \cap [t^{-1}t] \neq \emptyset$. This is equivalent to t being regular. □

Proposition 6.2.9. *Let S be a semigroup such that $\ell^1(S)$ is biflat. Suppose that:*

(i) *$R(S)$ is a subsemigroup of S;*

(ii) *$tS = St \ (t \in N(S))$;*

(iii) *$N(S)$ is finite.*

Then S is regular.

Proof. Assume towards a contradiction that $N(S) \neq \emptyset$ and take $s_1 \in N(S)$. A biflat Banach algebra is essential and so $S = S^2$. By (i) either $s_1 \in S \cdot N(S)$ or $s_1 \in N(S) \cdot S$. By (ii) this means that there exists $s_2 \in N(S)$ and elements $u_1, v_1 \in S$ with $s_1 = s_2 u_1 = v_1 s_2$.

Continuing like this we obtain a sequence $\{s_1, s_2, \ldots\} \subset N(S)$ such that for each $n \in \mathbb{N}$ there are elements $u_n, v_n \in S$ with

$$s_n = s_{n+1} u_n = v_n s_{n+1} \,.$$

By induction for each $1 \leq k < n$ we have

$$s_k = s_n u_{n-1} \cdots u_k = v_k \cdots v_{n-1} s_n \,.$$

Fix $n \geq 2$. Assume towards a contradiction that $s_n \in \{s_1, \ldots, s_{n-1}\}$ so that $s_n = s_k$ for some $k \in \{1, \ldots, n-1\}$. Then we have

$$s_k = s_k u_{n-1} \cdots u_k = v_k \cdots v_{n-1} s_k \,.$$

Hence $[s_k s_k^{-1}] \neq \emptyset$ and $[s_k^{-1} s_k] \neq \emptyset$. By Lemma 6.2.8, s_k is regular. This is a contradiction and so the elements $s_1, \ldots, s_n$ are all distinct. This is true for each $n \in N$, and so $N(S)$ is infinite, contradicting (iii). Therefore $N(S) = \emptyset$ and S is regular. □

Corollary 6.2.10. *Let S be a commutative semigroup such that $\ell^1(S)$ is biflat. Suppose that $N(S)$ is finite. Then S is regular.* □

Remark 6.2.11. After this work was completed we learned that Grønbæk and Habibian ([17]) have obtained similar results to ours. The authors in [17] give a different proof of our main theorem for commutative semigroups and use this to characterize the biflatness of $\ell^1(S)$ for a commutative semigroup S. Their article also studies the biflatness of Banach algebras graded over semilattices.

6.3 Left projectivity of the module $\ell^1(S)$

Let S be a semigroup. In this section we prove some results about the left projectivity of $\ell^1(S)$. Trivially, if $\ell^1(S)$ has a right identity, then $\ell^1(S)$ is left projective.

We first consider the semigroup $\mathbb{N}_\wedge$. Since $\mathbb{N}_\wedge$ is not uniformly locally finite, $\ell^1(\mathbb{N}_\wedge)$ is not biflat. The Banach algebra $\ell^1(\mathbb{N}_\wedge)$ has a bounded approximate identity and so $\ell^1(\mathbb{N}_\wedge)$ is a flat left $\ell^1(\mathbb{N}_\wedge)$-module. We shall prove that $\ell^1(\mathbb{N}_\wedge)$ is not a projective left $\ell^1(\mathbb{N}_\wedge)$-module.

Let E be a Banach space, and let $z = \sum_{i=1}^{\infty} \delta_i \otimes x_i \in \ell^1(\mathbb{N}_\wedge) \widehat{\otimes} E$. Then we have

$$\delta_n \cdot z = \sum_{i=1}^{n-1} \delta_i \otimes x_i + \delta_n \otimes \left(\sum_{i=n}^{\infty} x_i \right) \quad (n \in \mathbb{N}_\wedge)\,.$$

Lemma 6.3.1. *Let E be a Banach space, and let $\rho : \ell^1(\mathbb{N}_\wedge) \to \ell^1(\mathbb{N}_\wedge) \widehat{\otimes} E$ be a left $\ell^1(\mathbb{N}_\wedge)$-module morphism. Then there exists a sequence $\{x_i : i \in \mathbb{N}_\wedge\} \subset E$ such that*

$$\rho(\delta_n) = \sum_{i=1}^{n-1} \delta_i \otimes (x_i - x_{i+1}) + \delta_n \otimes x_n \quad (n \in \mathbb{N}_\wedge)\,. \tag{6.8}$$

Proof. Set $A = \ell^1(\mathbb{N}_\wedge)$, and for a subset $T \subset \mathbb{N}_\wedge$ set $A_T = \ell^1(T)$. We identify $A_T \widehat{\otimes} E$ with a closed complemented subspace of $A \widehat{\otimes} E$, namely the space of functions whose support is contained in T.

We define the sequence $\{x_i : i \in \mathbb{N}_\wedge\} \subset E$ by

$$x_i = (\lambda_i \otimes I_E)(\rho(\delta_i)) \quad (i \in \mathbb{N}_\wedge)\,.$$

We shall prove by induction that (6.8) holds.

For each $n \in \mathbb{N}$, we have $\rho(\delta_n) = \delta_n \cdot \rho(\delta_n) \in A_{[1,n]} \widehat{\otimes} E$. Hence $\rho(\delta_1) = \delta_1 \otimes x_1$.

Now let $n \geq 2$ and assume that

$$\rho(\delta_n) = \sum_{i=1}^{n-1} \delta_i \otimes (x_i - x_{i+1}) + \delta_n \otimes x_n\,.$$

We can write $\rho(\delta_{n+1}) = \sum_{i=1}^{n} \delta_i \otimes y_i + \delta_{n+1} \otimes x_{n+1}$ $(y_i \in E)$. The equation $\rho(\delta_n) = \delta_n \cdot \rho(\delta_{n+1})$ gives

$$\sum_{i=1}^{n-1} \delta_i \otimes (x_i - x_{i+1}) + \delta_n \otimes x_n = \sum_{i=1}^{n-1} \delta_i \otimes y_i + \delta_n \otimes (y_n + x_{n+1})\,.$$

For each $i \in \mathbb{N}_{n-1}$ applying the operator $\lambda_i \otimes I_E$ to this equation gives $y_i = x_i - x_{i+1}$, as required. □

For $n \in \mathbb{N}_\wedge$ we denote by φ_n the character on $\mathbb{N}_\wedge$ given by

$$\varphi_n(a) = \sum_{i=n}^{\infty} a_i \quad \left(a = \sum_{i \in \mathbb{N}} a_i \delta_i \in \ell^1(\mathbb{N}_\wedge) \right)\,.$$

Theorem 6.3.2. *The Banach algebra $\ell^1(\mathbb{N}_\wedge)$ is not left projective.*

Proof. We set $A = \ell^1(\mathbb{N}_\wedge)$.

Assume towards a contradiction that A is left projective. Then there exists $\rho \in {}_A\mathcal{B}(A, A)$ with $\pi \circ \rho = I_A$. Let $(a_i) \subset A$ be the sequence corresponding to ρ given by Lemma 6.3.1. From the equation $\delta_n = \pi(\rho(\delta_n))$, we obtain $\varphi_n(a_n) = 1 \;\; (n \in \mathbb{N}_\wedge)$.

Now fix $N \in \mathbb{N}$. Set $k_0 = 1$ and choose $k_1 > 1$ such that $\left\|\chi_{[k_1,\infty)} a_1\right\|_1 < 1/2$. Then we have

$$\|a_1 - a_{k_1}\|_1 \geq \left\|\chi_{[k_1,\infty)}(a_1 - a_{k_1})\right\|_1 \geq \varphi_{k_1}(a_{k_1}) - \left\|\chi_{[k_1,\infty)} a_1\right\|_1 > 1/2\,.$$

Next choose $k_2 > k_1$ such that $\left\|\chi_{[k_2,\infty)} a_{k_1}\right\|_1 < 1/2$. Again, we have

$$\|a_{k_1} - a_{k_2}\|_1 \geq \left\|\chi_{[k_2,\infty)}(a_{k_1} - a_{k_2})\right\|_1 \geq \varphi_{k_2}(a_{k_2}) - \left\|\chi_{[k_1,\infty)} a_{k_1}\right\|_1 > 1/2\,.$$

Continuing in this way we obtain an increasing sequence sequence $k_1 < \cdots < k_N$ such that $\left\|a_{k_i} - a_{k_{i+1}}\right\|_1 > 1/2 \;\; (i = 0, \ldots, N-1)$. But now

$$\|\rho\| > \|\rho(\delta_{k_N})\|_\pi = \sum_{i=1}^{k_N - 1} \|a_i - a_{i+1}\|_1 + \|a_{k_N}\|_1 \geq \sum_{i=0}^{N-1} \left\|a_{k_i} - a_{k_{i+1}}\right\|_1 > N/2\,.$$

This holds for each $N \in \mathbb{N}$, the required contradiction. □

The idea for the following proof came from [11, Theorem 10]. The proof below that τ is a morphism is very similar to part of [11, Lemma 9].

Theorem 6.3.3. *Let S be a semilattice such that $\ell^1(S)$ is projective in $\ell^1(S)$-**mod**. Let $s_1 < s_2 < \cdots$ be an increasing chain in S. Then there exists $s \in S$ with $s_i \leq s \;\; (i \in \mathbb{N})$.*

Proof. Assume towards a contradiction that the theorem is false. Then we can find an unbounded increasing sequence $s_1 < s_2 < \cdots$ in S. The map $\theta : \mathbb{N}_\wedge \to S$, $i \mapsto s_i$ is a homomorphism.

Now define $\tau : S \to \mathbb{N}_\wedge$ by

$$\tau(s) = \begin{cases} \max\{i : s_i \leq s\} \\ 1 \text{ if } \forall i,\;\; s_i \not\leq s \end{cases} \qquad (s \in S)\,.$$

By assumption there is no element s such that $s_i \leq s$ for all $i \in \mathbb{N}$, so τ is well defined. We shall show that τ is a homomorphism. Let $s, t \in S$.

Case 1 $\tau(s) \wedge \tau(t) = 1$. We have either $\forall i,\; s_i \not\leq s$ or $\forall i,\; s_i \not\leq t$. Assume that $\tau(st) = n > 1$. Then $s_n = s_n st$. Multiplying by s on the right we get $s_n s = s_n sts = s_n st = s_n$. Similarly $s_n t = s_n$. This is a contradiction, and therefore $\tau(st) = 1$.

Case 2 $\tau(s) = n, \tau(t) = m > 1$. Then $s_n \leq s$, $s_{n+1} \nleq s$ and $s_m \leq t$, $s_{m+1} \nleq t$. We may suppose $n \wedge m = n$. From the equations $s_n = s_n s$, $s_n = s_n s_m$ and $s_m = s_m t$, we get $s_n t = s_n st = s_n s_m st = s_n s_m s = s_n s = s_n$, so that $s_n \leq st$. Now assume that $s_{n+1} \leq st$. Then $s_{n+1} = s_{n+1} st$, multiplying by s we get $s_{n+1} s = s_{n+1} sts = s_{n+1}$, so that $s_{n+1} \leq st$, a contradiction. Therefore $s_{n+1} \nleq st$ and $\tau(st) = n = \tau(s) \wedge \tau(t)$.

We see that τ is a homomorphism, and that $\tau \circ \theta = I_{\mathbb{N}_\wedge}$. There are induced Banach algebra homomorphisms $\theta : \ell^1(\mathbb{N}_\wedge) \to \ell^1(S)$ and $\tau : \ell^1(S) \to \ell^1(\mathbb{N}_\wedge)$ such that $\tau \circ \theta = I_{\ell^1(\mathbb{N}_\wedge)}$. By Proposition 2.3.5 $\ell^1(\mathbb{N}_\wedge)$ must be left projective. This contradiction of Theorem 6.3.2 completes the proof. □

Corollary 6.3.4. *Let S be a Clifford semigroup such that $\ell^1(S)$ is projective in $\ell^1(S)$-***mod***. Let $s_1 < s_2 < \cdots$ be an increasing chain in $E(S)$. Then there exists $s \in E(S)$ with $s_i \leq s$ $(i \in \mathbb{N})$.*

Proof. Note that $E(S)$ is a semilattice.

For any $u \in S$, we have $uu^{-1} \in E(S)$ and $u \in G_{uu^{-1}}$. Let $s, t \in S$. Since $G_p G_q \subset G_{pq}$ $(p, q \in E(S))$, we have $st \in G_{ss^{-1}tt^{-1}}$. Since $st \in G_{stt^{-1}s^{-1}}$ and the groups G_p $(p \in E(S))$ are disjoint, we must have $ss^{-1}tt^{-1} = stt^{-1}s^{-1}$. Hence the map $\theta : s \mapsto ss^{-1}$, $S \to E(S)$ is a semigroup homomorphism which extends to a homomorphism of Banach algebras $\theta : \ell^1(S) \to \ell^1(E(S))$. The restriction of θ to $\ell^1(E(S))$ is the identity map, hence $\ell^1(E(S))$ is a retract of $\ell^1(S)$. Since $\ell^1(S)$ is left projective by Proposition 2.3.5, $\ell^1(E(S))$ is left projective, and hence the result follows from Theorem 6.3.3. □

Chapter 7

Injectivity of $\ell^1(S)$

Let S be a semigroup. In this chapter we shall study the injectivity of the right $\ell^1(S)$-module $\ell^1(S)$. Suppose that S is weakly right cancellative. Then, by Proposition 1.7.3 $c_0(S) \in \ell^1(S)$-**mod**, and we can identify the dual right $\ell^1(S)$-module $c_0(S)'$ with $\ell^1(S)$. Hence for weakly right cancellative semigroups, the injectivity of $\ell^1(S)$ in **mod**-$\ell^1(S)$ is the same as flatness of $c_0(S)$ in $\ell^1(S)$-**mod**. For such semigroups we also study the stronger property of the projectivity of $c_0(S)$ in $\ell^1(S)$-**mod**.

We recall the answers in the case where $S = G$ is a group.

Theorem 7.0.5 ([8, Theorem 4.9, Theorem 3.1])**.** *Let G be a group. Then:*

(i) *$\ell^1(G)$ is injective in* **mod**-*$\ell^1(S)$ if and only if G is amenable.*

(ii) *$c_0(G)$ is projective in $\ell^1(G)$*-**mod** *if and only if G is finite.* □

7.1 Injectivity and amenability

The following theorem is a generalization of [8, Theorem 3.1].

Theorem 7.1.1. *Let S be a semigroup such that $\ell^1(S)$ is injective in* **mod**-*$\ell^1(S)$. Then S is a left-amenable semigroup and $\ell^1(S)$ has a left identity.*

Proof. That S is left-amenable follows in the same way as the proof of Theorem 3.4.1. That $\ell^1(S)$ has a left identity is Corollary 2.2.8(i). □

Let A be a Banach algebra with a left identity e_A. Then e_A is an identity for A if and only if $\{a \in A : aA = \{0\}\} = \{0\}$ i.e., A is a faithful right A-module.

Example 7.1.2. Let S be the *right-zero semigroup.* The product is given by

$$st = t \quad (s, t \in S)\,.$$

The Banach algebra $\ell^1(S)$ belongs to the class of Banach algebras of the form $A_\varphi(X)$ defined in Example 2.2.4. By Proposition 2.2.5, $\ell^1(S)$ is right injective if and only if $|S| = 2$. Since in this case $\ell^1(S)$ is finite dimensional, this is equivalent to the predual module $c_0(S) = \ell^\infty(S)$ being projective in $\ell^1(S)$-**mod**.

Let S_2 be the right-zero semigroup on 2 elements. We note that $\ell^1(S_2)$ is not amenable. Indeed S_2 is left-amenable but not right-amenable; further, S_2 has a left identity, but $\ell^1(S_2)$ does not have a right identity. Hence the plausible conjecture that $\ell^1(S)$ is injective in **mod**-$\ell^1(S)$ only if $\ell^1(S)$ is amenable is false.

Lemma 7.1.3. *Let S be a semigroup such that $\ell^1(S)$ is injective in* **mod**-$\ell^1(S)$*. Let $t \in S$ be a left cancellable element. Then there exists $a_t \in \ell^1(S)$ with*

$$a_t \star \delta_t \star e_A = e_A$$

for each left identity $e_A \in \ell^1(S)$.

Proof. Set $A = \ell^1(S)$. There is a map

$$T : tS \to S : ts \mapsto s$$

which extends to a bounded linear operator $T : \ell^1(tS) \to A$. Set $U = T \circ P$, where $P : A \to \ell^1(tS)$ is a projection. Then U satisfies

$$U(b \star a) = U(b) \star a \quad (b \in \ell^1(tS),\, a \in A)\,.$$

By Proposition 2.2.6, there exists $a_t \in A$ with $U(b) = a_t \star b \ \ (b \in \ell^1(tS))$. In particular, $e_A = U(\delta_t \star e_A) = a_t \star \delta_t \star e_A$. □

Theorem 7.1.4. *Let S be a cancellative semigroup. Then $\ell^1(S)$ is injective in* **mod**-$\ell^1(S)$ *if and only if S is an amenable group.*

Proof. Sufficiency is obvious; we shall prove necessity.

Suppose that $\ell^1(S)$ is an injective right module. By Theorem 7.1.1, the Banach algebra $\ell^1(S)$ has a left identity. By Lemma 1.7.2, S has an identity e_S. By Lemma 7.1.3, for each $s \in S$, there exists $a_s \in \ell^1(S)$ with $a_s \star \delta_s = \delta_{e_S}$. It follows that there is an element $s^{-1} \in S$ with $s^{-1}s = e_S$. From the equation $ss^{-1}s = e_S s$ and right cancellativity we have $ss^{-1} = e_S$. Therefore S is a group. It is amenable by Theorem 7.1.1. □

Corollary 7.1.5. *The Banach algebra $\ell^1(\mathbb{N})$ is not injective in* **mod**-$\ell^1(\mathbb{N})$*, equivalently $c_0(\mathbb{N})$ is not flat in $\ell^1(\mathbb{N})$-***mod***.* □

7.2 Flatness of the predual module $c_0(S)$

Now we restrict to the class of weakly cancellative semigroups, and study the flatness of the left $\ell^1(S)$-module $c_0(S)$.

For the next two lemmas, we suppose that S is a semigroup, and that E is a Banach space. For a subset $T \subset S$, we set $X_T = \ell^1(T) \widehat{\otimes} E$. For an element $t \in S$, and $F \in \ell^1(S)$-**mod**, we set ${}^{t\perp}F = {}^{\{\delta_t\}\perp}F$.

Lemma 7.2.1. *For each $t \in S$, we have*

$$({}^{t\perp}X_S)^0 = X'_S \cdot t\,.$$

Proof. Let $t \in S$. The inclusion $X'_S \cdot t \subset ({}^{t\perp}X_S)^0$ is clear.

Take $\lambda \in ({}^{t\perp}X_S)^0$ and $u, v \in S$ with $tu = tv$. Then for each $x \in E$ we have

$$\delta_u \otimes x - \delta_v \otimes x \in {}^{t\perp}X_S\,.$$

Hence, under the identification $X'_S = \ell^\infty(S, E')$ we have

$$0 = \langle \delta_u \otimes x - \delta_v \otimes x, \lambda \rangle = \langle x, \lambda(u) - \lambda(v) \rangle \quad (x \in E)\,.$$

It follows that $\lambda(u) = \lambda(v)$. Hence, for each $s \in S$, λ is constant on the set $[t^{-1}s]$. Therefore we can define

$$\varphi(s) = \begin{cases} \lambda(u), & \text{if there exists } u \in [t^{-1}s]\,, \\ 0, & \text{if } [t^{-1}s] = \emptyset\,, \end{cases} \qquad (s \in S)\,.$$

Then $\varphi \in X'_S$ and $\lambda = \varphi \cdot t$. □

For a subset $T \subset S$ and an element $t \in S$, we define the following subset of T:

$$G(T,t) = \bigcup_{\{s \in S : |[t^{-1}s] \cap T| \geq 2\}} [t^{-1}s] \cap T = \{u \in T : \exists v \in T, v \neq u, tu = tv\}\,.$$

The complement of $G(T,t)$ in T is perhaps simpler to describe: we have

$$T \setminus G(T,t) = \{s \in T : [t^{-1}(ts)] \cap T = \{s\}\}\,.$$

For example, if t is a left cancellable element, then $G(T,t) = \emptyset$.

For a subset $T \subset S$, we identify X_T with the closed, complemented subspace of X_S consisting of functions on S whose support is contained in T. We can then identify X''_T with X_T^{00} in X''_S.

Lemma 7.2.2. *For each $t \in S$ and $T \subset S$, we have:*

(i) $t \cdot X''_S \subset (X_{tS})''$;

(ii) ${}^{t\perp}(X''_T) \subset X''_{G(T,t)}$.

Proof. (i) Let $t \in S$ and $\varphi \in (X_{tS})^0$. Then $\varphi \cdot t = 0$, and so for each $\Lambda \in X''_S$, we have

$$\langle \varphi, t \cdot \Lambda \rangle = \langle \varphi \cdot t, \Lambda \rangle = 0\,.$$

Therefore $t \cdot \Lambda \in X_{tS}^{00} = X_{tS}''$.

(ii) Let $t \in S$ and $T \subset S$. Take $z = \sum_{s\in S} \delta_s \otimes x_s \in {}^{t\perp}X_T$. The equation $t \cdot z = 0$ gives

$$\sum_{u\in[t^{-1}s]\cap T} x_u = 0 \quad (s \in S)\,.$$

Take $u \in \text{supp } z$. Then $u \in [t^{-1}(tu)] \cap T$. Hence $|[t^{-1}(tu)] \cap T| \geq 2$, and $z \in X_{G(T,t)}$. We have proved that ${}^{t\perp}X_T \subset X_{G(T,t)}$. Now by Lemma 7.2.1 we have

$$(X_{G(T,t)})^0 \subset ({}^{t\perp}X_T)^0 = X_T' \cdot t\,.$$

Hence ${}^{t\perp}(X_T'') = (X_T' \cdot t)^0 \subset (X_{G(T,t)})^{00} = X_{G(T,t)}''$. □

Theorem 7.2.3. *Let S be a weakly right cancellative semigroup such that, for each $N \in \mathbb{N}$, there exist elements $(s_1, r_1, t_1), \ldots, (s_N, r_N, t_N)$ in $S\times S\times S$ with the following properties:*

(i) $s_n \in Sr_n \setminus St_nr_n$ $(n \in \mathbb{N}_N)$;

(ii) *the sets* $[s_1r_1^{-1}], \ldots, [s_Nr_N^{-1}]$ *are pairwise disjoint;*

(iii) *the sets* $G(r_1S^\flat, t_1), \ldots, G(r_NS^\flat, t_N)$ *are pairwise disjoint.*

*Then $c_0(S)$ is not flat in $\ell^1(S)$-**mod**.*

Proof. We set $E = c_0(S)$, and for a subset $T \subset S^\flat$, we set $A_T = \ell^1(T)$.

Assume towards a contradiction that E is flat in $\ell^1(S)$-**mod**. Then there exists a left $\ell^1(S)$-module morphism $\rho : E \to (A_{S^\flat} \widehat{\otimes} E)''$ with $\pi'' \circ \rho = i_E$. Fix $N \in \mathbb{N}$, and let $(s_1, r_1, t_1), \ldots, (s_N, r_N, t_N) \in S \times S \times S$ be the elements given by the hypothesis. For each $n \in \mathbb{N}_N$, set

$$x_n = r_n \cdot \lambda_{s_n} = \chi_{[s_nr_n^{-1}]}\,.$$

By Lemma 7.2.2(i) $\rho(x_n) \in (A_{r_nS^\flat} \widehat{\otimes} E)''$. Since $s_n \notin St_nr_n$, we have

$$t_n \cdot \rho(x_n) = \rho(t_n \cdot x_n) = \rho(\chi_{[s_n(t_nr_n)^{-1}]}) = 0\,,$$

Hence by Lemma 7.2.2(ii) $\rho(x_n) \in (A_{G(r_nS^\flat,t_n)} \widehat{\otimes} E)''$.

Set $\Phi = \rho\left(\sum_{n=1}^N x_n\right) \in (A_{S^\flat} \widehat{\otimes} E)''$. By (ii) $\left\|\sum_{n=1}^N x_n\right\|_\infty = 1$. Hence there is a net (z_α) in $(A_{S^\flat} \widehat{\otimes} E)_{[\|\rho\|]}$ such that $\lim_\alpha z_\alpha = \Phi$ in the weak-$*$ topology. For each $n \in \mathbb{N}_N$, let $P_n : A_{S^\flat} \widehat{\otimes} E \to A_{G(r_nS^\flat,t_n)} \widehat{\otimes} E$ be a projection. By (iii) we have $P_n''(\rho(x_m)) = 0$ $(n \neq m)$. Hence $\lim_\alpha P_n''(z_\alpha) = P_n''(\Phi) = \rho(x_n)$ in the weak-$*$ topology.

For each $i \in \mathbb{N}_N$, by (i) we can pick $u_n \in [s_nr_n^{-1}]$. For large enough α, we have

$$|\langle \Pi(\delta_{u_n}), P_n(z_\alpha)\rangle - \langle \Pi(\delta_{u_n}), \rho(x_n)\rangle| < 1/2\,.$$

Now for each $n \in \mathbb{N}_N$, we have

$$\langle \Pi(\delta_{u_n}), \rho(x_n) \rangle = \langle \delta_{u_n}, \pi'' \circ \rho(x_n) \rangle = \langle x_n, \delta_{u_n} \rangle = 1 \,.$$

Hence, for large enough α, we have

$$|\langle \Pi(\delta_{u_n}), P_n(z_\alpha) \rangle| > 1/2 \,,$$

and so $\|P_n(z_\alpha)\|_\pi \geq 1/2$. But now using (iii), for sufficiently large α,

$$\|\rho\| \geq \|z_\alpha\|_\pi \geq \sum_{n=1}^{N} \|P_n(z_\alpha)\|_\pi \geq N/2 \,.$$

This holds for each $N \in \mathbb{N}$, the required contradiction. □

Lemma 7.2.4. *Let S be a semigroup, and let $p, q \in E(S)$ with $p < q$. Then $pS \subset qS$ and*

$$G(qS, p) = (qS \setminus pS) \cup p(qS \setminus pS)$$

is a disjoint union of sets.

Proof. It is clear that $pS \subset qS$ and $(qS \setminus pS) \cap p(qS \setminus pS) = \emptyset$.

Let $\pi_1 : S \times S \to S$ be the projection onto the first coordinate. Consider the sets

$$\mathcal{G} = \{(u, v) \in qS \times qS : u \neq v, pu = pv\}$$

and

$$\mathcal{F} = \{(u, pu) : u \in qS \setminus pS\} \cup \{(pu, u) : u \in qS \setminus pS\} \,.$$

We make the identifications

$$\pi_1(\mathcal{G}) = G(qS, p) \quad \text{and} \quad \pi_1(\mathcal{F}) = (qS \setminus pS) \cup p(qS \setminus pS) \,.$$

Since $\mathcal{F} \subset \mathcal{G}$ we have $\pi_1(\mathcal{F}) \subset \pi_1(\mathcal{G})$. Now take $u \in \pi_1(\mathcal{G})$ with $u = \pi_1((u, v))$ for some $(u, v) \in \mathcal{G}$. If $u \notin pS$, then $(u, pu) \in \mathcal{F}$. If $u \in pS$, then $v \in qS \setminus pS$ and $(u, v) = (pv, v) \in \mathcal{F}$. In either case $u \in \pi_1(\mathcal{F})$. □

Theorem 7.2.5. *Let S be a weakly right cancellative semigroup. Suppose that there exists an infinite chain of idempotents*

$$r_1 > s_1 > t_1 > \cdots > r_n > s_n > t_n > r_{n+1} > s_{n+1} > t_{n+1} > \cdots$$

*such that for each $n \in \mathbb{N}$, $t_n(r_n S \setminus t_n S) \cap r_{n+1} S = \emptyset$. Then $c_0(S)$ is not flat in $\ell^1(S)$-**mod**.*

Proof. We shall apply Theorem 7.2.3; we verify clauses (i)-(iii).

Clearly $s_n \in Sr_n \setminus St_n = Sr_n \setminus St_nr_n$ $(n \in \mathbb{N})$, so that clause (i) holds.

Take $n < m$. Assume towards a contradiction that there exists $u \in [s_nr_n^{-1}] \cap [s_mr_m^{-1}]$. Then $ur_n = s_n$ and $ur_m = s_m$. Multiplying the first of these equations on the right by r_m gives $ur_m = r_m$. Hence $r_m = s_m$, which is a contradiction. Therefore clause (ii) holds.

For each $k \leq m$ we have

$$t_kS \supset r_mS\,.$$

Hence we have

$$(r_nS \setminus t_nS) \cap G(r_mS, t_m) \subset (S \setminus t_nS) \cap r_mS = \emptyset\,.$$

Using Lemma 7.2.4, we have

$$\begin{aligned} G(r_nS, t_n) \cap G(r_mS, t_m) &= t_n(r_nS \setminus t_nS) \cap G(r_mS, t_m) \\ &\subset t_n(r_nS \setminus t_nS) \cap r_mS \\ &\subset t_n(r_nS \setminus t_nS) \cap r_{n+1}S \\ &= \emptyset \quad \text{(by hypothesis)}\,. \end{aligned}$$

Therefore condition (iii) of Theorem 7.2.3 also holds, and hence $c_0(S)$ is not flat in $\ell^1(S)$-**mod**. □

Example 7.2.6. We give some examples of semigroups which satisfy the hypothesis of Theorem 7.2.5.

(i) Let $S = \mathbb{N}_\vee$. The canonical partial order on S is the reverse of the natural order on $\mathbb{N}$. We set

$$r_n = 3n - 2, \quad s_n = 3n - 1, \quad t_n = 3n \quad (n \in \mathbb{N})\,.$$

For any $n \in S$, we have $nS = [n, \infty)$. Hence for each $n \in \mathbb{N}$ we have $t_n(r_nS \setminus t_nS) = 3n[3n-2, 3n-1] = \{3n\}$, which is disjoint from the set $r_{n+1}S = [3n+1, \infty)$. Therefore $c_0(\mathbb{N}_\vee)$ is not flat in $\ell^1(\mathbb{N}_\vee)$-**mod**.

(ii) Let B be the *bicyclic semigroup*. Then $B = \mathbb{N}_0 \times \mathbb{N}_0$ with the multiplication

$$(m, n)(p, q) = (m - n + \max\{n, p\}, q - p + \max\{n, p\}) \quad ((m, n), (p, q) \in B)\,.$$

We set

$$r_n = (3n-2, 3n-2), \quad s_n = (3n-1, 3n-1), \quad t_n = (3n, 3n) \quad (n \in \mathbb{N})\,.$$

For any $(m, n) \in B$, we have $(m, n)B = [m, \infty) \times \mathbb{N}_0$. Hence for each $n \in \mathbb{N}$ we have

$$t_n(r_nB \setminus t_nB) = (3n, 3n)([1, 3n-1] \times \mathbb{N}_0) = \{3n\} \times \mathbb{N}_0\,,$$

which is disjoint from the set $r_{n+1}B = [3n+1, \infty) \times \mathbb{N}_0$. Therefore $c_0(B)$ is not flat in $\ell^1(B)$-**mod**.

7.3 Projectivity of the predual module $c_0(S)$

For a weakly right cancellative semigroup S we now consider when $c_0(S)$ has the stronger property of being projective in $\ell^1(S)$-**mod**.

Lemma 7.3.1. *Let S be an infinite, weakly right cancellative semigroup such that, for every finite set $F \subset S$, there exists $r \in S$ with $rS^\flat \cap F = \emptyset$. Suppose that $c_0(S)$ is projective in $\ell^1(S)$-**mod** with splitting morphism $\rho : c_0(S) \to \ell^1(S^\flat) \widehat{\otimes} c_0(S)$. Then for each $N \in \mathbb{N}$, there exist elements $x_1, \dots, x_N$ in $c_0(S)$ and a partition $\{F_1, \dots, F_N\}$ of S with the properties:*

(i) $\left\| \sum_{i=1}^N x_i \right\|_\infty = 1$,

(ii) $\|\rho(x_i)\|_\pi \geq 1$ *for each* $i \in \mathbb{N}_N$, *and,*

(iii) $\|\chi_{F_i}\rho(x_i) - \rho(x_i)\|_\pi < 1/3^i$ *for each* $i \in \mathbb{N}_N$.

Proof. We set $E = c_0(S)$, and for a subset $T \subset S$, we set $A_T = \ell^1(T)$.

Fix $N \in \mathbb{N}$. To begin, choose $r_1, t_1 \in S$ with $[t_1 r_1^{-1}] \neq \emptyset$ and set

$$x_1 = r_1 \cdot \lambda_{t_1} = \chi_{[t_1 r_1^{-1}]} \,.$$

Then $\rho(x_1) \in A_{r_1 S^\flat} \widehat{\otimes} E = \ell^1(r_1 S^\flat, E)$, and $1 = \|x_1\|_\infty = \|\pi \circ \rho(x_1)\|_\infty \leq \|\rho(x_1)\|_\pi$. Take a finite set $F_1 \subset r_1 S^\flat$ with $\|\chi_{F_1}\rho(x_1) - \rho(x_1)\|_\pi < 1/3$.

Now suppose that $x_1, \dots, x_k$ and $\{F_1, \dots, F_k\}$ are already constructed. Choose $r_{k+1} \in S$ with $r_{k+1}S^\flat \cap \bigcup_{i=1}^k F_i = \emptyset$. Set

$$G = \bigcup_{i=1}^{k} [t_i r_i^{-1}] \quad \text{and} \quad H = \bigcup_{s \in G} [(s r_{k+1}) r_{k+1}^{-1}] \,.$$

The set H is finite, so we can choose an element u in the complement. Set

$$t_{k+1} = u r_{k+1} \quad \text{and} \quad x_{k+1} = r_{k+1} \cdot \lambda_{t_{k+1}} = \chi_{[t_{k+1} r_{k+1}^{-1}]} \,.$$

Since $u \in [t_{k+1} r_{k+1}^{-1}]$ we have $\|\rho(x_{k+1})\|_\pi \geq 1$.

We shall show that the set $[t_{k+1} r_{k+1}^{-1}]$ is disjoint from G. Assume towards a contradiction that there exists $v \in [t_{k+1} r_{k+1}^{-1}] \cap G$. Then $v r_{k+1} = t_{k+1} = u r_{k+1}$, and so $u \in [(v r_{k+1}) r_{k+1}^{-1}] \subset H$. This is a contradiction, and therefore $[t_{k+1} r_{k+1}^{-1}] \cap G = \emptyset$, whence $\left\| \sum_{i=1}^{k+1} x_i \right\|_\infty = 1$. We have $\rho(x_{k+1}) \in A_{r_{k+1}S^\flat} \widehat{\otimes} E$. Take a finite set $F_{k+1} \subset r_{k+1}S^\flat$ with $\left\| \chi_{F_{k+1}}\rho(x_{k+1}) - \rho(x_{k+1}) \right\|_\pi < 1/3^{k+1}$.

In the final stage we may take $F_N = S \setminus \bigcup_{i=1}^{N-1} F_i$, so that the sets (F_i) form a partition of S. □

Theorem 7.3.2. *Let S be an infinite, weakly right cancellative semigroup. Suppose that $c_0(S)$ is projective in $\ell^1(S)$-**mod**. Then there exists a finite set $F \subset S$ such that, for each $r \in S$,*

$$rS^\flat \cap F \neq \emptyset \,.$$

Proof. We set $A_{S^\flat} = \ell^1(S^\flat)$ and $E = c_0(S)$.

Since E is projective in $\ell^1(S)$-**mod**, there exists a left $\ell^1(S)$-module morphism $\rho : E \to A_{S^\flat} \widehat{\otimes} E$ with $\pi \circ \rho = I_E$. Assume towards a contradiction that the condition is not satisfied, so that we can apply Lemma 7.3.1. Fix $N \in \mathbb{N}$, and let $x_1, \ldots, x_N$ and $\{F_1, \ldots, F_N\}$ be the elements corresponding to ρ given by Lemma 7.3.1.

Firstly, for each $m \in \mathbb{N}_N$ we have

$$\left\| \chi_{F_m}\rho\left(\sum_{i\neq m} x_i\right)\right\|_\pi \leq \sum_{i\neq m} \|\chi_{F_m}\rho(x_i)\|_\pi \leq \sum_{i\neq m} \left\|\chi_{S\setminus F_i}\rho(x_i)\right\|_\pi \leq \sum_{i\neq m}\frac{1}{3^i}$$

and $\|\chi_{F_m}\rho(x_m)\|_\pi \geq 1 - \frac{1}{3^m}$. Hence

$$\left\| \chi_{F_m}\rho\left(\sum_{i=1}^N x_i\right)\right\|_\pi \geq 1 - \frac{1}{3^m} - \sum_{i\neq m}\frac{1}{3^i} \geq 1 - \left(\frac{1}{1-1/3} - 1\right) = \frac{1}{2}\,,$$

and so,

$$\|\rho\| \geq \left\|\rho\left(\sum_{i=1}^N x_i\right)\right\|_\pi = \left\|\chi_{F_1}\rho\left(\sum_{i=1}^N x_i\right)\right\|_\pi + \cdots + \left\|\chi_{F_N}\rho\left(\sum_{i=1}^N x_i\right)\right\|_\pi \geq \frac{N}{2}\,.$$

This holds for each $N \in \mathbb{N}$, the required contradiction. □

Corollary 7.3.3. *Let S be a weakly right cancellative semigroup. Suppose that $c_0(S)$ is projective in $\ell^1(S)$-**mod**. Then the set*

$$\bigcup_{t\in S}[tt^{-1}]$$

is finite. Hence the set $E(S)$ is also finite.

Proof. Let F be the finite set given by Theorem 7.3.2. Take $t \in S$ and $u \in [tt^{-1}]$. Then either $t \in F$ or $f = ts$ for some $s \in S$ and $f \in F$. In the latter case we have $uf = uts = ts = f$, and so $u \in [ff^{-1}]$. Hence

$$\bigcup_{t\in S}[tt^{-1}] \subset \bigcup_{f\in F}[ff^{-1}]\,,$$

and so the set $\bigcup_{t\in S}[tt^{-1}]$ is finite.

For each $p \in E(S)$ we have $p \in [pp^{-1}]$, hence the set $E(S)$ is finite. □

Lemma 7.3.4. *Let S be an infinite, weakly right cancellative semigroup. Suppose that $c_0(S)$ is projective in $\ell^1(S)$-**mod**. Then there exists an element $t \in S$ such that, for every finite set $F \subset S$, there exists $r \in S \setminus F$ with $[tr^{-1}] \neq \emptyset$.*

Proof. Assume towards a contradiction that the conclusion is false. Then, for every $t \in S$, there exists a finite set $F(t) \subset S$ such that $[tr^{-1}] = \emptyset$ for all $r \in S \setminus F(t)$.

Let F be the finite set given by Theorem 7.3.2. Take $s \in S \setminus F$. Then $su = f$ for some $f \in F$ and $u \in S$. Since $s \in [fu^{-1}]$ it must be that $u \in F(f)$, and hence

$$S \subset \bigcup_{f \in F} \bigcup_{u \in F(f)} [fu^{-1}] \cup F\,.$$

But the set on the right-hand side is finite, and so S is finite. This is a contradiction. Therefore the conclusion holds. □

We now suppose that S is a weakly cancellative semigroup. Using the above lemma we can now apply an argument similar to that of the group case [8, Theorem 3.1], to show that S must be finite.

Theorem 7.3.5. *Let S be a weakly cancellative semigroup such that $c_0(S)$ is projective in $\ell^1(S)$-***mod***. Then S is finite.*

Proof. Let $\rho : c_0(S) \to \ell^1(S^\flat) \widehat{\otimes} c_0(S)$ be a left $\ell^1(S)$-module morphism with $\pi \circ \rho = I_{c_0(S)}$.

Assume towards a contradiction that S is infinite. Let $t \in S$ be the element specified in Lemma 7.3.4. Fix $N \in \mathbb{N}$, and take a finite set $F \subset S^\flat$ with $\|\chi_F \rho(\lambda_t) - \rho(\lambda_t)\|_\pi < 1/N$. We shall construct elements $r_1, \ldots, r_N \in S$ with the following properties:

(i) the sets $[tr_1^{-1}], \ldots, [tr_N^{-1}]$ are pairwise disjoint and non-empty, and

(ii) the sets $r_1F, \ldots, r_NF$ are pairwise disjoint.

To begin choose any $r_1 \in S$ with $[tr_1^{-1}] \neq \emptyset$. Now suppose that $r_1, \ldots, r_k$ are already constructed. Set

$$X(k) = \bigcup_{i=1}^{k} \bigcup_{f,g \in F} [(r_i f)g^{-1}], \quad Y(k) = \bigcup_{i=1}^{k} [tr_i^{-1}], \quad Z(k) = \bigcup_{u \in Y(k)} [u^{-1}t]\,.$$

Since the sets $X(k)$ and $Z(k)$ are finite, we can use Lemma 7.3.4 to choose an element $r_{k+1} \in S \setminus X(k) \cup Z(k)$ with $[tr_{k+1}^{-1}] \neq \emptyset$. We now show that clauses (i) and (ii) are satisfied.

Take $1 \leq i < j \leq N$. Assume that there exists $u \in [tr_i^{-1}] \cap [tr_j^{-1}]$. Then $ur_j = t$, and so $r_j \in [u^{-1}t]$ for $u \in [tr_i^{-1}] \subset Y(j-1)$. Hence $r_j \in Z(j-1)$, which is a contradiction, giving clause (i). Next assume that there exists $v \in r_iF \cap r_jF$ so that $r_if = r_jg$ for some $f, g \in F$. But then $r_j \in [(r_if)g^{-1}] \subset X(j-1)$, which is a contradiction, and so clause (ii) holds.

For each $i \in \mathbb{N}_N$, since $[tr_i^{-1}] \neq \emptyset$, we have $\|r_i \cdot \rho(\lambda_t)\|_\pi \geq 1$ and we have the norm estimate

$$\|r_i \cdot \chi_F \rho(\lambda_t)\|_\pi \geq \|r_i \cdot \rho(\lambda_t)\|_\pi - \|r_i \cdot (\chi_F \rho(\lambda_t) - \rho(\lambda_t))\|_\pi \geq 1 - 1/N.$$

Now, we have

$$\begin{aligned}\|\rho\| &\geq \left\|\rho\left(\sum_{i=1}^{N} r_i \cdot \lambda_t\right)\right\|_\pi \geq \left\|\sum_{i=1}^{N} r_i \cdot \chi_F \rho(\lambda_t)\right\|_\pi - \left\|\sum_{i=1}^{N} r_i \cdot (\rho(\lambda_t) - \chi_F \rho(\lambda_t))\right\|_\pi \\ &= \sum_{i=1}^{N} \|r_i \cdot \chi_F \rho(\lambda_t)\|_\pi - \left\|\sum_{i=1}^{N} r_i \cdot (\rho(\lambda_t) - \chi_F \rho(\lambda_t))\right\|_\pi \\ &\geq N(1 - 1/N) - N/N = N - 2\,.\end{aligned}$$

This holds for each $N \in \mathbb{N}$, the required contradiction. Therefore S is finite. □

Let S be a finite inverse semigroup. Then $\ell^1(S)$ is contractible [11, Theorem 8], and so every $E \in \ell^1(S)$-**mod** is projective. Hence we have the following.

Theorem 7.3.6. *Let S be a weakly cancellative inverse semigroup. Then $c_0(S)$ is projective in $\ell^1(S)$-**mod** if and only if S is finite.* □

7.4 Non-injectivity of $\ell^1(S)$ for an infinite free semilattice

Let X be a set, let $P_X = \mathcal{P}(X)$ be the power set of X, and let F_X the set of all finite subsets of X. Then P_X is a semilattice with the multiplication

$$st = s \cup t \quad (s, t \in P_X)\,,$$

and F_X is a subsemilattice of P_X called the *free semilattice* over X. The empty set is the identity of P_X, which we denote by e. For $s, t \in P_X$, we have

$$[st^{-1}] = [t^{-1}s] = \{u : ut = s\} = \begin{cases} \emptyset \text{ if } t \not\subset s \\ \{s \setminus u : u \subset t\} \text{ if } t \subset s \end{cases}.$$

For $t \subset s \in F_X$ we have $|[t^{-1}s]| = 2^{|t|}$, and hence F_X is weakly cancellative. For each $t \in P_X$, we have $[tt^{-1}] = \{u : u \subset t\}$. The canonical partial order on P_X is given by

$$s \leq t \Longleftrightarrow t \subset s \quad (s, t \in P_X)\,.$$

Take $r, t \in P_X$ with $r \not\leq t$ i.e., $t \setminus r \neq \emptyset$. Take $u \in rP_X$. We can write $u = r \cup s$ where $s \cap r = \emptyset$. Set

$$v = \begin{cases} u \cup (t \setminus u) \text{ if } t \setminus u \neq \emptyset \\ u \setminus (s \cap t) \text{ if } s \cap t \neq \emptyset \end{cases}.$$

The condition $t \setminus r \neq \emptyset$ ensures that one of these cases must occur. Then $u \neq v$ and $t \cup u = t \cup v$. Hence $u \in G(rP_X, t)$ and $G(rP_X, t) = rP_X$. Hence for $r_1 \not\leq t_1$ and

$r_2 \not\leq t_2$ we have $G(r_1P_X, t_1) \cap G(r_2P_X, t_2) = r_1r_2P_X$. Therefore Theorem 7.2.5 gives no information about the injectivity of the module $\ell^1(P_X)$.

We shall prove that, if X is an infinite set, then $\ell^1(P_X)$ and $\ell^1(F_X)$ are not right injective Banach algebras. Clearly P_X and F_X only depend on the cardinality of X. For $N \in \mathbb{N}$, we set $P_N = P_{\mathbb{N}_N}$. The Banach algebra $\ell^1(P_N)$ is contractible and has a unique diagonal d with $\|d\| = 5^N$ [15, Example 1.6]. By Proposition 1.3.11, $\ell^1(P_N)$ is 5^N-injective in **mod**-$\ell^1(P_N)$. Let C_N be the minimum C such that $\ell^1(P_N)$ is C-injective in **mod**-$\ell^1(P_N)$. We shall find the exact value of C_N and show that $C_N \to \infty$ as $N \to \infty$. From this we can deduce the result.

Clearly $\ell^1(P_N)$ is C-injective in **mod**-$\ell^1(P_N)$ if and only if $\ell^\infty(P_N)$ is C-projective in $\ell^1(P_N)$-**mod**. We prefer to work with the latter formulation of the problem.

7.4.1 Preliminaries

We first collect together some combinatorial results that we need. These follow easily from the binomial theorem. For $m, n \in \mathbb{N}_0$. The symbol $\binom{n}{m}$ denotes the number of m-subsets of an n-set. We set $\binom{n}{m} = 0$ if $m > n$.

Lemma 7.4.1. *Let X be a finite set.*

(i)

$$\sum_{s\subset X}(-1)^{|s|} = \begin{cases} 1 \text{ if } X = \emptyset \\ 0 \text{ if } X \neq \emptyset \end{cases}.$$

(ii) *Let $t \subset X$. Then*

$$\sum_{s\subset X}(-1)^{|s\cap t|} = \begin{cases} 2^{|X|} \text{ if } t = \emptyset \\ 0 \text{ if } t \neq \emptyset \end{cases}.$$

(iii) *Let $t_1, t_2 \subset X$. Then*

$$\sum_{s\subset X}(-1)^{|s\cap t_1|+|s\cap t_2|} = \begin{cases} 2^{|X|} \text{ if } t_1 = t_2 \\ 0 \text{ if } t_1 \neq t_2 \end{cases}.$$

Proof. (i) The case where $X = \emptyset$ is clear. Suppose that $X \neq \emptyset$. Then we have

$$\sum_{s\subset X}(-1)^{|s|} = \sum_{k=0}^{|X|}\sum_{\{s\subset X, |s|=k\}}(-1)^{|s|} = \sum_{k=0}^{|X|}\binom{|X|}{k}(-1)^k = (1-1)^{|X|} = 0\,.$$

(ii) We have, using part (i):

$$\sum_{s\subset X}(-1)^{|s\cap t|} = \sum_{s\subset t}\sum_{r\subset X\setminus t}(-1)^{|(s\cup r)\cap t|} = \sum_{s\subset t}2^{|X\setminus t|}(-1)^{|s|} = \begin{cases} 2^{|X|} \text{ if } t = \emptyset \\ 0 \text{ if } t \neq \emptyset \end{cases}.$$

(iii) Set $t = t_1 \,\Delta\, t_2$, the symmetric difference of t_1 and t_2. For each $s \subset X$ we have

$$(-1)^{|s\cap t_1|+|s\cap t_2|} = (-1)^{|s\cap t|+2|s\cap t_1\cap t_2|} = (-1)^{|s\cap t|},$$

and so the result follows from part (ii). □

Lemma 7.4.2. *Take $a, b, N \in \mathbb{N}$ with $b \le a < N$. Then*

$$\sum_{k=0}^{N-1} \binom{N-a}{k-b} (-1)^k = \begin{cases} (-1)^{N-1} \text{ if } a = b \\ 0 \text{ if } b < a \end{cases} .$$

Proof. We denote the sum by Σ. We have

$$\Sigma = \sum_{k=-b}^{N-1-b} \binom{N-a}{k} (-1)^{k+b} = \sum_{k=0}^{\min\{N-a, N-b-1\}} \binom{N-a}{k} (-1)^{k+b} .$$

Consider first the case where $a = b$. Then

$$\Sigma = (-1)^a \sum_{k=0}^{N-a-1} \binom{N-a}{k} (-1)^k = (-1)^a (0 - (-1)^{N-a}) = (-1)^{N-1} .$$

Now suppose that $b < a$. Then we have

$$\Sigma = (-1)^b \sum_{k=0}^{N-a} \binom{N-a}{k} (-1)^k = 0 .$$ □

7.4.2 $\ell^\infty(P_N)$ is a $2^{N/2}$-left projective module

Let S be a finite semigroup, let E be a Banach space, and let $\rho \in \mathcal{B}(\ell^\infty(S), \ell^1(S) \widehat{\otimes} E)$. Then ρ is a left $\ell^1(S)$-module morphism if and only if

$$r \cdot \rho(\lambda_t) = \sum_{u \in [tr^{-1}]} \rho(\lambda_u) \quad (r, t \in S) . \tag{7.1}$$

For each $t \in S$, we can write

$$\rho(\lambda_t) = \sum_{s \in S} \delta_s \otimes x_s^t , \tag{7.2}$$

where $x_s^t \in E$ $(s \in S)$. Then equation (7.1) holds if and only if

$$\sum_{u \in [r^{-1}s]} x_u^t = \sum_{u \in [tr^{-1}]} x_s^u \quad (r, s, t \in S) . \tag{7.3}$$

Theorem 7.4.3. *Let $N \in \mathbb{N}$, and let E be a Banach space. Let*

$$\rho \in \mathcal{B}(\ell^\infty(P_N), \ell^1(P_N) \widehat{\otimes} E) .$$

Then ρ is a left $\ell^1(P_N)$-module morphism if and only if there exists a set

$$\{x_s : s \in P_N\} \subset E$$

such that

$$\rho(\lambda_t) = \sum_{s \in P_N} \delta_s \otimes (-1)^{|t|+|s|} x_{t \cap s} \quad (t \in P_N) . \tag{7.4}$$

Proof. First suppose that (7.4) defines a map $\rho : \ell^\infty(P_N) \to \ell^1(P_N) \widehat{\otimes} E$. We shall show that equation (7.3) holds. In the notation of (7.2) we have $x_s^t = (-1)^{|t|+|s|} x_{t\cap s}$. We denote the expressions on the left- and right-hand sides of equation (7.3) by $L(r,s,t)$ and $R(r,s,t)$, respectively. For $r,s,t \in P_N$ set

$$F(r,s,t) = (-1)^{|t|+|s|} \sum_{u\subset r} (-1)^{|u|} x_{(t\cap s)\setminus u}\,.$$

Fix $r,s \in P_N$ with $[r^{-1}s] \neq \emptyset$. Then it is easily checked that for all $t \in P_N$, $L(r,s,t) = F(r,s,t)$. Now fix $t,r \in P_N$ with $[tr^{-1}] \neq \emptyset$. Again it is easily checked that for all $s \in P_N$, $R(r,s,t) = F(r,s,t)$.

Now fix $r,t,s \in P_N$. Clearly (7.3) holds in the case where $[r^{-1}s] \neq \emptyset$ and $[tr^{-1}] \neq \emptyset$ since $L(r,s,t) = R(r,s,t) = F(r,s,t)$. The case where $[r^{-1}s] = [tr^{-1}] = \emptyset$ is also clear since $L(r,s,t) = R(r,s,t) = 0$. Suppose that $[r^{-1}s] \neq \emptyset$ and $[tr^{-1}] = \emptyset$. The latter condition implies that $R(r,s,t) = 0$ and that $r \cap (\mathbb{N}_N \setminus t) \neq \emptyset$. Then we have

$$\begin{aligned} L(r,s,t) = F(r,s,t) &= (-1)^{|t|+|s|} \sum_{u\subset r\cap t\cap s} (-1)^{|u|} \left(\sum_{v\subset r\cap(\mathbb{N}_N\setminus t\cap s)} (-1)^{|v|} \right) x_{t\cap(s\setminus v)} \\ &= 0 \quad \text{(by Lemma 7.4.1(i))}\,, \end{aligned}$$

and so $L(r,s,t) = R(r,s,t)$ in this case. Exactly the same argument works in the case where $[r^{-1}s] = \emptyset$ and $[tr^{-1}] \neq \emptyset$. Therefore by (7.3) ρ is a left $\ell^1(P_N)$-module morphism.

Now we shall prove the converse. Take ρ as in (7.2). We first prove that for each $t \in P_N$, we have

$$x_v^t = x_t^v \quad (v \subset t)\,. \tag{7.5}$$

The proof is by induction on $|t \setminus v| = |t| - |v|$. The result is clear if $|t \setminus v| = 0$ i.e., if $v = t$. Suppose that $x_v^t = x_t^v$ for all $v \subset t$ with $|t\setminus v| \leq k$. Take $v \subset t$ with $|t \setminus v| = k+1$. Set $s = t$, $r = t\setminus v$ in (7.3) to obtain

$$\sum_{u\subset t\setminus v} x_{t\setminus u}^t = \sum_{u\subset t\setminus v} x_t^{t\setminus u}\,.$$

If $u \subsetneq t\setminus v$, then $|t\setminus(t\setminus u)| = |u| < |t\setminus v| = k+1$. Hence for $u \subsetneq t\setminus v$ we have $|t\setminus(t\setminus u)| \leq k$. By the induction hypothesis all such terms in the above sum cancel leaving $x_v^t = x_t^v$, and the result is proved.

Now we shall show that

$$x_v^t = (-1)^{|t|+|v|} x_{t\cap v}^{t\cap v} \quad (t,v \in P_N)\,. \tag{7.6}$$

The proof is by induction on $|t|$.

We first consider the case where $|t| = 0$ i.e., $t = e$. We have to show that $x_v^e = (-1)^{|v|}x_e^e$ for all $v \in P_N$. We shall prove this by induction on $|v|$. The result is clear if $|v| = 0$. Suppose that $x_v^e = (-1)^{|v|}x_e^e$ for all v with $|v| \leq k$. Take $v \in P_N$ with $|v| = k+1$. Set $r = s = v$ in (7.3). Then $R(v, v, e) = 0$ and we have

$$\begin{aligned} x_v^e &= -\sum_{u \subsetneq v} x_u^e = -\sum_{k=0}^{|v|-1} \sum_{\{u \subset v, |u|=k\}} x_u^e = -\sum_{k=0}^{|v|-1} \binom{|v|}{k} (-1)^k x_e^e \\ &= -(0 - (-1)^{|v|})x_e^e = (-1)^{|v|}x_e^e\,, \end{aligned}$$

which proves that (7.6) holds in the case where $t = e$.

Next suppose that $x_v^t = (-1)^{|t|+|v|}x_{t\cap v}^{t\cap v}$ for all $t \in P_N$ with $|t| \leq k$ and all $v \in P_N$. Take $t \in P_N$ with $|t| = k+1$. We have to show that $x_v^t = (-1)^{|t|+|v|}x_{t\cap v}^{t\cap v}$ for all $v \in P_N$. We shall use induction on $|v|$. By (7.5) the result holds if $|v| = 0$. Indeed, for any $v \subset t$ by (7.5) and the induction hypothesis on t, we have

$$x_v^t = x_t^v = (-1)^{|v|+|t|}x_v^v\,,$$

and so the result holds in this case. Now suppose that the result holds for all v with $|v| \leq k$. Take $v \in P_N$ with $|v| = k+1$, and set $t_0 = v \cap t$. Set $r = s = v$ in (7.3). We may suppose that $v \not\subset t$, so that $R(v, v, t) = 0$. Then we have

$$\begin{aligned} x_v^t &= -\sum_{u \subsetneq v} x_u^t = -\sum_{k=0}^{|v|-1} \sum_{\{u \subset v, |u|=k\}} x_u^t = -\sum_{k=0}^{|v|-1} \sum_{t_1 \subset t_0} \sum_{\{u \subset v, |u|=k, u\cap t = t_1\}} x_u^t \\ &= -\sum_{k=0}^{|v|-1} \sum_{t_1 \subset t_0} \binom{|v| - |t_0|}{k - |t_1|} (-1)^{|t|+k} x_{t_1}^{t_1} = -(-1)^{|t|+|v|-1}x_{t_0}^{t_0} = (-1)^{|t|+|v|}x_{t_0}^{t_0}\,, \end{aligned}$$

where, in the last line, we have used Lemma 7.4.2. Therefore (7.6) holds for all $t, v \in P_N$.

Finally we set $x_v = x_v^v$ $(v \in P_N)$, so that, for each $t \in P_N$, we have

$$\rho(\lambda_t) = \sum_{s \in P_N} \delta_s \otimes x_s^t = \sum_{s \in P_N} \delta_s \otimes (-1)^{|s|+|t|}x_{s\cap t}^{s\cap t} = \sum_{s \in P_N} \delta_s \otimes (-1)^{|s|+|t|}x_{s\cap t}\,,$$

as required. □

Let $N \in \mathbb{N}$, and set $E = \ell^\infty(P_N)$. Let ρ be as in (7.4). We shall impose the identity $\pi \circ \rho = I_E$ and see what restrictions this puts on the elements $\{x_r\}$. The identity $\pi \circ \rho = I_E$ holds if and only if

$$\pi \circ \rho(\lambda_t) = \lambda_t \quad (t \in P_N)\,. \tag{7.7}$$

For each $s \in P_N$ we can write

$$x_s = \sum_{r \in P_N} \beta_r^s \lambda_r\,, \tag{7.8}$$

where $\{\beta_r^s : r \in P_N\} \subset \mathbb{C}$. For each $t \in P_N$, equation (7.7) becomes

$$\begin{aligned}\lambda_t &= \sum_{s\in P_N} s \cdot x_s^t = \sum_{s\in P_N} (-1)^{|t|+|s|} s \cdot x_{t\cap s} = \sum_{s\in S} (-1)^{|t|+|s|} \sum_{r\in P_N} \beta_r^{t\cap s} \chi_{[rs^{-1}]} \\ &= \sum_{u\in P_N} \left(\sum_{\{(r,s):u\in[rs^{-1}]\}} (-1)^{|t|+|s|} \beta_r^{t\cap s} \right) \lambda_u \,.\end{aligned}$$

This holds if and only if

$$\begin{aligned}\delta_{tu} &= \sum_{\{(r,s):u\in[rs^{-1}]\}} (-1)^{|t|+|s|} \beta_r^{t\cap s} = \sum_{r\in uS} \sum_{s\in[u^{-1}r]} (-1)^{|t|+|s|} \beta_r^{t\cap s} \\ &= (-1)^{|t|} \sum_{t\supset u} \sum_{s\subset u} (-1)^{|r\setminus s|} \beta_r^{t\cap(r\setminus s)} \quad (u \in P_N)\,.\end{aligned}$$

This can be rewritten as

$$(-1)^{|t|}\delta_{tu} = \sum_{r\supset u} (-1)^{|r|} \sum_{s\subset u} (-1)^{|s|} \beta_r^{t\cap(r\setminus s)} \quad (u \in P_N)\,. \tag{7.9}$$

Surprisingly, this equation automatically holds for certain $t, u \in P_N$.

Lemma 7.4.4. *Equation* (7.9) *holds for all* $t, u \in P_N$ *with* $u \setminus t \neq \emptyset$.

Proof. For $r \supset u$ the inner sum on the right hand side of (7.9) is

$$\begin{aligned}\sum_{s\subset u} (-1)^{|s|} \beta_r^{t\cap(r\setminus s)} &= \sum_{s_1\subset u\cap t} \sum_{s_2\subset u\setminus t} (-1)^{|s_1|+|s_2|} \beta_r^{t\cap(r\setminus s_1)} \\ &= \begin{cases} 0 \ \text{if} \ u\setminus t \neq \emptyset \\ \sum_{s_1\subset u\cap t} (-1)^{|s_1|} \beta_r^{t\cap(r\setminus s_1)} \ \text{if} \ u\setminus t = \emptyset \end{cases} \quad \text{(by Lemma 7.4.1(i))}\,.\end{aligned}$$

Hence both sides of (7.9) are equal to 0. □

Corollary 7.4.5. *The identity* $\pi \circ \rho = I_E$ *holds if and only if equation* (7.9) *holds for all* $t, u \in P_N$ *with* $u \subset t$. □

We require an expression for $\|\rho\|$. It is convenient to consider

$$x = \sum_{t\in P_N} (-1)^{|t|} \gamma_t \lambda_t \in \ell^\infty(P_N)\,,$$

where $\{\gamma_t : t \in P_N\} \subset \mathbb{C}$. We have

$$\rho(x) = \sum_{s\in P_N} \delta_s \otimes (-1)^{|s|} \sum_{t\in P_N} \gamma_t x_{t\cap s}\,.$$

By the identification $\ell^1(P_N) \widehat{\otimes} \ell^\infty(P_N) = \ell^1(P_N, \ell^\infty(P_N))$, we have

$$\|\rho(x)\|_\pi = \sum_{s\in P_N} \left\| \sum_{t\in P_N} \gamma_t x_{t\cap s} \right\|\,. \tag{7.10}$$

Theorem 7.4.6. *Let $N \in \mathbb{N}$. Then $\ell^\infty(P_N)$ is $2^{N/2}$-projective in $\ell^1(P_N)$-***mod**.

Proof. We define a map $\rho : \ell^\infty(P_N) \to \ell^1(P_N) \widehat{\otimes} \ell^\infty(P_N)$ via (7.4) and (7.8) by setting

$$\beta_r^s = \frac{1}{|P_N|}(-1)^{|r|+|s|} \quad (r, s \in P_N)\,.$$

By Theorem 7.4.3, ρ is a left $\ell^1(P_N)$-module morphism. We check that (7.9) holds. Take $t, u \in P_N$ with $u \subset t$. We shall denote the right-hand side of (7.9) by $\Sigma(t,u)$. We have

$$|P_N|\,\Sigma(t,u) = \sum_{r \supset u} \sum_{s \subset u} (-1)^{|s|+|t\cap(r\setminus s)|}\,.$$

For each r and s in the sum above, we have $s \subset r \cap t$, and so $|t \cap (r \setminus s)| = |t \cap r| - |s|$. This gives

$$\begin{aligned}
|P_N|\,\Sigma(t,u) &= \sum_{r \supset u} \sum_{s \subset u} (-1)^{|t\cap r|} = \sum_{r \supset u} 2^{|u|}(-1)^{|t \cap r|} = 2^{|u|} \sum_{r \subset \mathbb{N}_N \setminus u} (-1)^{|u| + |(t\setminus u)\cap r|} \\
&= 2^{|u|}(-1)^{|u|} 2^{N-|u|} \delta_{\emptyset (t \setminus u)} \quad \text{(by Lemma 7.4.1(ii))} \\
&= |P_N|\,(-1)^{|t|}\delta_{tu}\,.
\end{aligned}$$

Therefore (7.9) holds. By Corollary 7.4.5, we have $\pi \circ \rho = I_{\ell^\infty(P_N)}$.

Next we estimate $\|\rho\|$. For $x = \sum_{t \in P_N} (-1)^{|t|}\gamma_t \lambda_t \in \ell^\infty(P_N)$ and $s \in P_N$ we first calculate

$$\begin{aligned}
\left\| \sum_{t \in P_N} \gamma_t x_{t \cap s} \right\| &= \left\| \sum_{r \in P_N} \left(\sum_{t \in P_N} \gamma_t \beta_r^{t \cap s} \right) \lambda_r \right\| = \frac{1}{|P_N|} \left\| \sum_{r \in P_N} \left(\sum_{t \in P_N} \gamma_t (-1)^{|t\cap s| + |r|} \right) \lambda_r \right\| \\
&= \frac{1}{|P_N|} \left| \sum_{t \in P_N} \gamma_t (-1)^{|t \cap s|} \right|\,.
\end{aligned}$$

By (7.10) we have

$$\begin{aligned}
|P_N|\,\|\rho(x)\| &= \sum_{s \in P_N} \left| \sum_{t \in P_N} \gamma_t(-1)^{|t\cap s|} \right| \\
&\le |P_N|^{1/2} \left(\sum_{s \in P_N} \left| \sum_{t \in P_N} \gamma_t (-1)^{|t \cap s|} \right|^2 \right)^{1/2} \quad \text{(by Hölder's inequality)} \\
&= |P_N|^{1/2} \left[\sum_{s \in P_N} \left(\sum_{t \in P_N} \gamma_t (-1)^{|t\cap s|} \right) \left(\sum_{t \in P_N} \overline{\gamma}_t (-1)^{|t \cap s|} \right) \right]^{1/2} \\
&= |P_N|^{1/2} \left[\sum_{t, r \in P_N} \left(\sum_{s \in P_N} (-1)^{|t \cap s| + |r \cap s|} \right) \gamma_t \overline{\gamma_r} \right]^{1/2} \\
&= |P_N|^{1/2} \left[|P_N| \sum_{t \in P_N} |\gamma_t|^2 \right]^{1/2} \quad \text{(by Lemma 7.4.1(iii))} \\
&= |P_N|\,\|x\|_2 \le |P_N|^{3/2}\,\|x\|_\infty\,.
\end{aligned}$$

Therefore $\|\rho\| \le |P_N|^{1/2} = 2^{N/2}$, and the result is proved. □

7.4.3 $2^{N/2}$ is the best possible constant of projectivity

Let $N \in \mathbb{N}$. In this section we shall prove that $\ell^\infty(P_{2N})$ is not C-left projective in $\ell^1(P_{2N})$-**mod** for any $1 < C < 2^N$.

We now fix throughout this section $N \in \mathbb{N}$ and a left $\ell^1(P_N)$-morphism

$$\rho : \ell^\infty(P_{2N}) \to \ell^1(P_{2N}) \widehat{\otimes} \ell^\infty(P_{2N}) \quad \text{with} \quad \pi \circ \rho = I_{\ell^\infty(P_{2N})} . \tag{7.11}$$

We shall prove that $\|\rho\| \geq 2^N$. Let $\{x_t : t \in P_{2N}\} \subset \ell^\infty(P_{2N})$ be the set of elements corresponding to ρ given in Theorem 7.4.3.

We set

$$p_n = \{2n-1, 2n\} \in P_{2N} \quad (n \in \mathbb{N}_N) .$$

We inductively define sets of complex numbers $\{\{\gamma_t^{2n} : t \in P_{2n}\} : n \in \mathbb{N}_N\}$ by setting

$$\gamma_e^2 = \gamma_1^2 = \gamma_2^2 = 1, \quad \gamma_{p_1}^2 = -1 ,$$

and for each $2 \leq n \leq N$ setting

$$\gamma_t^{2n} = \begin{cases} \gamma_{t \setminus p_n}^{2(n-1)} & \text{if } p_n \not\subset t \\ -\gamma_{t \setminus p_n}^{2(n-1)} & \text{if } p_n \subset t \end{cases} \qquad (t \in P_{2n}) .$$

Now we define a vector $y \in \ell^\infty(P_{2N})_{[1]}$ by

$$y = \sum_{t \in P_{2N}} \gamma_t^{2N} (-1)^{|t|} \lambda_t . \tag{7.12}$$

It is convenient to introduce the following notation. For $n \in \mathbb{N}_N$ and $s \in P_{2n}$, we set

$$\Lambda(s; 2n) = \sum_{t \in P_{2n}} \gamma_t^{2n} x_{t \cap s} .$$

Recall from (7.10) that

$$\|\rho(y)\| = \sum_{s \in P_{2N}} \|\Lambda(s; 2N)\| . \tag{7.13}$$

We shall prove that $\|\rho(y)\| \geq 2^N$. The proof consists of the following steps:

(i) First we derive a recurrence relation between the $\Lambda(s; 2n)$'s.

(ii) Then we prove a formula for $\Lambda(s; 2N)$.

(iii) Next we estimate 'part' of (7.13).

(iv) Finally we add up the 'parts' to get an estimate for (7.13).

A recurrence relation

Consider the vector space $\mathrm{lin}\{(s, x_s) : s \in P_{2N}\}$ of formal linear combinations. We define an action $*$ of $\ell^1(P_{2N})$ on this space by setting

$$(s, x_s) * \delta_t = (s \cup t, x_{s\cup t}) \quad (s, t \in P_{2N}) .$$

In order to simplify our notation we shall identify x_s with (s, x_s), and write $x_s * \delta_t = x_{s\cup t}$. For example, with this notation we have

$$\Lambda(s; 2n) * (\alpha\delta_{t_1} + \beta\delta_{t_2}) = \alpha \sum_{t\in P_{2n}} \gamma_t^{2n} x_{(t\cap s)\cup t_1} + \beta \sum_{t\in P_{2n}} \gamma_t^{2n} x_{(t\cap s)\cup t_2} .$$

Lemma 7.4.7. *Let $2 \le n \le N$, and let $s \in P_{2n}$. If $p_n \not\subset s$, then*

$$\Lambda(s; 2n) = 2\Lambda(s \setminus p_n; 2(n-1)) . \tag{7.14}$$

If $p_n \subset s$, then

$$\Lambda(s; 2n) = \Lambda(s \setminus p_n; 2(n-1)) * (\delta_e + \delta_{\{2n-1\}} + \delta_{\{2n\}} - \delta_{p_n}) . \tag{7.15}$$

Proof. We have

$$\begin{aligned}\Lambda(s; 2n) = &\sum_{t\in P_{2(n-1)}} \gamma_{t\cup p_n}^{2n} x_{(t\cup p_n)\cap s} + \sum_{t\in P_{2(n-1)}} \gamma_{t\cup\{2n-1\}}^{2n} x_{(t\cup\{2n-1\})\cap s} \\ &+ \sum_{t\in P_{2(n-1)}} \gamma_{t\cup\{2n\}}^{2n} x_{(t\cup\{2n\})\cap s} + \sum_{t\in P_{2(n-1)}} \gamma_t^{2n} x_{t\cap s} \\ = &- \sum_{t\in P_{2(n-1)}} \gamma_t^{2(n-1)} x_{(t\cup p_n)\cap s} + \sum_{t\in P_{2(n-1)}} \gamma_t^{2(n-1)} x_{(t\cup\{2n-1\})\cap s} \\ &+ \sum_{t\in P_{2(n-1)}} \gamma_t^{2(n-1)} x_{(t\cup\{2n\})\cap s} + \sum_{t\in P_{2(n-1)}} \gamma_t^{2n} x_{t\cap s} .\end{aligned}$$

If $p_n \subset s$, then we have the result stated. If $s \cap p_n = \{2n\}, \{2n-1\}$ or e, then the first term cancels with the second, third or fourth term respectively, giving the result in this case. □

A formula for $\Lambda(s; 2N)$

For $s \in P_{2N}$ we set

$$F(s) = \sum_{t\subset s} (-1)^{|t|} \delta_t \in \ell^1(P_{2N}) \quad \text{and} \quad \Gamma(s) = \sum_{t\subset s} (-1)^{|t|} x_t \in \ell^\infty(P_{2N}) .$$

Lemma 7.4.8. *Let $s, t \in P_{2N}$ with $s \cap t = \emptyset$. Then*

$$\Gamma(s) * F(t) = \Gamma(s \cup t) .$$

Proof. We calculate

$$\Gamma(s \cup t) = \sum_{r \subset s \cup t} (-1)^{|r|} x_r = \sum_{t_0 \subset t} \sum_{s_0 \subset s} (-1)^{|t_0|+|s_0|} x_{t_0 \cup s_0} = \sum_{t_0 \subset t} (-1)^{|t_0|} \Gamma(s) * \delta_{t_0}$$
$$= \Gamma(s) * \left(\sum_{t_0 \subset t} (-1)^{|t_0|} \delta_{t_0} \right) = \Gamma(s) * F(t) \,. \qquad \square$$

For $s \in P_{2N}$ we set

$$Y(s) = \{ n \in \mathbb{N}_N : p_n \subset s \} \,,$$

and, for each $0 \le k \le N$, we let $Y_k(s)$ denote the family of k-subsets of $Y(s)$.

Lemma 7.4.9. *Let $s \in P_{2N}$. Then*

$$\Lambda(s; 2N) = \sum_{k=1}^{N} 2^{N-k} (-1)^k \sum_{\{n_1, \dots, n_k\} \in Y_k(s)} \Gamma(p_{n_1} \cup \cdots \cup p_{n_k}) + 2^N x_e \,. \tag{7.16}$$

Proof. For $n \in \mathbb{N}_N$, let $Q(n)$ denote the statement that for all $s \in P_{2n}$, the following equation holds:

$$\Lambda(s; 2n) = \sum_{k=1}^{n} 2^{n-k} (-1)^k \sum_{\{n_1, \dots, n_k\} \in Y_k(s)} \Gamma(p_{n_1} \cup \cdots \cup p_{n_k}) + 2^n x_e \,. \tag{7.17}$$

We shall prove by induction on n that $Q(n)$ is true for each $n \in \mathbb{N}_N$.

First consider the case where $n = 1$, so that $P_{2n} = \{e, \{1\}, \{2\}, p_1\}$. If $s = e, \{1\}$ or $\{2\}$, then it is easily checked that

$$\Lambda(s; 2) = 2x_e \,,$$

which agrees with (7.17) since $Y_1(s) = \emptyset$ for $s = e, \{1\}$, or $\{2\}$. A similar direct check shows that

$$\Lambda(p_1; 2) = \gamma_e^2 x_e + \gamma_{\{1\}}^2 x_{\{1\}} + \gamma_{\{2\}}^2 x_{\{2\}} + \gamma_{\{p_1\}}^2 x_{\{p_1\}} = -\Gamma(p_1) + 2x_e \,,$$

and so $Q(1)$ is true.

Now assume that $Q(n)$ holds for a fixed $n \in \mathbb{N}_{N-1}$. Take $s \in P_{2(n+1)}$. First suppose that $p_{n+1} \not\subset s$. Then by (7.14) we have

$$\begin{aligned}
\Lambda(s; 2(n+1)) &= 2\Lambda(s \setminus p_{n+1}; 2n) \\
&= 2 \left(\sum_{k=1}^{n} 2^{n-k} (-1)^k \sum_{\{n_1, \dots, n_k\} \in Y_k(s \setminus p_{n+1})} \Gamma(p_{n_1} \cup \cdots \cup p_{n_k}) + 2^n x_e \right) \\
&= \sum_{k=1}^{n+1} 2^{n+1-k} (-1)^k \sum_{\{n_1, \dots, n_k\} \in Y_k(s)} \Gamma(p_{n_1} \cup \cdots \cup p_{n_k}) + 2^{n+1} x_e \,.
\end{aligned}$$

The last line follows since $Y_k(s) = Y_k(s \setminus p_{n+1})$ $(k \in \mathbb{N}_n)$ and $Y_{n+1}(s) = \emptyset$.

Now suppose that $p_{n+1} \subset s$. By (7.15) and Lemma 7.4.8 we have

$$\begin{aligned}
&\Lambda(s; 2(n+1)) = 2\Lambda(s \setminus p_{n+1}; 2n) * (2\delta_e - F(p_{n+1})) \\
&= \left(\sum_{k=1}^{n} 2^{n-k}(-1)^k \sum_{\{n_1,\ldots,n_k\} \in Y_k(s \setminus p_{n+1})} \Gamma(p_{n_1} \cup \cdots \cup p_{n_k}) + 2^n x_e \right) * (2\delta_e - F(p_{n+1})) \\
&= \sum_{k=1}^{n} 2^{n+1-k}(-1)^k \sum_{\{n_1,\ldots,n_k\} \in Y_k(s \setminus p_{n+1})} \Gamma(p_{n_1} \cup \cdots \cup p_{n_k}) + 2^{n+1} x_e \\
&\quad + \sum_{k=1}^{n} 2^{n+1-(k+1)}(-1)^{k+1} \sum_{\{n_1,\ldots,n_k\} \in Y_k(s \setminus p_{n+1})} \Gamma(p_{n_1} \cup \cdots \cup p_{n_k} \cup p_{n+1}) - 2^n \Gamma(p_{n+1}) .
\end{aligned}$$

This last line is a partition of the sum in (7.17). Therefore $Q(n+1)$ is true.

Therefore by induction $Q(n)$ is true for all $n \in \mathbb{N}_N$. In particular $Q(N)$ is true. □

Norm estimates

Lemma 7.4.10. *Let $r \in P_{2N}$. Then*

$$\|\Gamma(r)\| \geq \frac{1}{2^{2N-|r|}} .$$

Proof. We define $x = \sum_{t \in P_{2N}} (-1)^{|t|} \gamma_t \lambda_t \in \ell^\infty(P_{2N})$ by

$$\gamma_t = \begin{cases} (-1)^{|t|} \text{ if } t \subset r \\ 0 \text{ otherwise} \end{cases} \qquad (t \in P_{2N}) .$$

First observe that for $s \in P_{2N}$ we have

$$\begin{aligned}
\sum_{t \subset r} \gamma_t x_{t \cap s} &= \sum_{s_1 \subset s \cap r} \left(\sum_{\{t \subset r : t \cap s = s_1\}} (-1)^{|t|} \right) x_{s_1} = \sum_{s_1 \subset s \cap r} \left(\sum_{t \subset r \setminus s} (-1)^{|s_1|+|t|} \right) x_{s_1} \\
&= \begin{cases} \sum_{s_1 \subset r} (-1)^{|s_1|} x_{s_1} \text{ if } r \subset s \\ 0 \text{ if } r \not\subset s \end{cases} \qquad \text{(by Lemma 7.4.1(i))} .
\end{aligned}$$

Since $1 = \|x\| = \|\pi \circ \rho(x)\| \leq \|\rho(x)\|$, by (7.10) we have

$$1 \leq \sum_{s \in P_{2N}} \left\| \sum_{t \subset r} \gamma_t x_{t \cap s} \right\| = \sum_{s \supset r} \left\| \sum_{s_1 \subset r} (-1)^{|s_1|} x_{s_1} \right\| = 2^{2N-|r|} \left\| \sum_{t \subset r} (-1)^{|t|} x_t \right\| ,$$

which gives the result. □

Lemma 7.4.11. *Let $s \in P_{2N}$. Then*

$$\sum_{\{n_1,\ldots,n_k\} \subset Y(s)} \|\Lambda(s \setminus p_{n_1} \cup \cdots \cup p_{n_k}; 2N)\| \geq \frac{1}{2^{N-|Y(s)|}} . \qquad (7.18)$$

Proof. Set $Y = Y(s)$ and $\Lambda(t) = \Lambda(t; 2N)$ $(t \in P_{2N})$. We define

$$\sum_{\{n_1,\dots,n_0\}\in Y_0(s)} \Gamma(p_{n_1} \cup \cdots \cup p_{n_0}) = x_e$$

so that (7.16) can be combined into a single sum.

By (7.16) we have

$$\sum_{\{n_1,\dots,n_k\}\subset Y} \|\Lambda(s \setminus p_{n_1} \cup \cdots \cup p_{n_k})\| = \|\Lambda(s)\| + \sum_{\substack{\{n_1,\dots,n_k\}\subset Y\\ \{n_1,\dots,n_k\}\neq\emptyset}} \|\Lambda(s \setminus p_{n_1} \cup \cdots \cup p_{n_k})\|$$

$$\geq \left\|2^{N-|Y|}\Gamma\left(\cup_{n\in Y} p_n\right)\right\| - \left\|\sum_{k=0}^{|Y|-1} 2^{N-k}(-1)^k \sum_{\{n_1,\dots,n_k\}\subset Y_k} \Gamma(p_{n_1} \cup \cdots \cup p_{n_k})\right\|$$

$$+ \sum_{\substack{\{n_1,\dots,n_k\}\subset Y\\ \{n_1,\dots,n_k\}\neq\emptyset}} \|\Lambda(s \setminus p_{n_1} \cup \cdots \cup p_{n_k})\| \quad \text{by } (7.16)\,.$$

By Lemma 7.4.10 we have

$$\left\|2^{N-|Y|}\Gamma(\cup_{n\in Y} p_n)\right\| \geq \frac{2^{N-|Y|}}{2^{2N-2|Y|}} = \frac{1}{2^{N-|Y|}}\,.$$

Hence the result will follow if we can show that

$$\left\|\sum_{k=0}^{|Y|-1} 2^{N-k}(-1)^k \sum_{\{n_1,\dots,n_k\}\subset Y_k} \Gamma(p_{n_1} \cup \cdots \cup p_{n_k})\right\| \leq \sum_{\substack{\{n_1,\dots,n_k\}\subset Y\\ \{n_1,\dots,n_k\}\neq\emptyset}} \|\Lambda(s \setminus p_{n_1} \cup \cdots \cup p_{n_k})\|\,.$$

This inequality follows from the identity

$$\sum_{k=0}^{|Y|-1} 2^{N-k}(-1)^k \sum_{\{n_1,\dots,n_k\}\subset Y_k} \Gamma(p_{n_1}\cup\cdots\cup p_{n_k}) = \sum_{l=1}^{|Y|}(-1)^l \sum_{\{n_1,\dots,n_l\}\in Y_l} \Lambda(s\setminus p_{n_1}\cup\cdots\cup p_{n_l})\,.$$

To see that this is true, take $\{n_1, \dots, n_k\} \subset Y$. The coefficient of $\Gamma(p_{n_1} \cup \cdots \cup p_{n_k})$ on the left-hand side is $2^{N-k}(-1)^k$, and on the right-hand side, using (7.16), it is

$$\sum_{l=1}^{|Y|-k} (-1)^l \binom{|Y|-k}{l} 2^{N-k}(-1)^k = 2^{N-k}(-1)^k\,,$$

which gives the result. □

Theorem 7.4.12. *Let $M \in \mathbb{N}$. Then $\ell^\infty(P_M)$ is $2^{M/2}$-projective in $\ell^1(P_M)$-**mod**, but not C-projective for any $1 < C < 2^{\lfloor M/2\rfloor}$.*

Proof. By Theorem 7.4.6, $\ell^\infty(P_M)$ is $2^{M/2}$-projective in $\ell^1(P_M)$-**mod**.

Suppose that $M = 2N$ for some $N \in \mathbb{N}$, and that $\ell^\infty(P_{2N})$ is C-projective in $\ell^1(P_{2N})$-**mod**. Let ρ be as in (7.11), and let y be as in (7.12).

We call a set $s \in P_{2N}$ *special* if, for each $n \in \mathbb{N}_N$, either $2n-1 \in s$ or $2n \in s$. Let $0 \leq k \leq N$. Then a set $s \in P_{2N}$ is a *special set of order k* if s is a special set and $|Y(s)| = k$. The collection of special sets of order k is denoted by $\mathcal{S}_k$. We have $|\mathcal{S}_k| = \binom{N}{k}2^{N-k}$. The important fact about special sets is that for every $t \in P_{2N}$ there exists a unique special set s with $t = s \setminus p_{n_1} \cup \cdots \cup p_{n_l}$ for some $\{n_1, \ldots, n_l\} \subset Y(s)$. Combining these facts with Lemma 7.4.11 gives

$$\begin{aligned}
\|\rho\| \geq \|\rho(y)\| &= \sum_{s\in P_{2N}} \|\Lambda(s; 2N)\| \quad \text{by (7.13)} \\
&= \sum_{k=0}^{N} \sum_{s\in\mathcal{S}_k} \sum_{\{n_1,\ldots,n_l\}\subset Y(s)} \|\Lambda(s \setminus p_{n_1} \cup \cdots \cup p_{n_l}; 2N)\| \\
&\geq \sum_{k=0}^{N} \binom{N}{k} 2^{N-k} \frac{1}{2^{N-k}} = 2^N .
\end{aligned}$$

Therefore $C \geq 2^N$, and the result is proved in the case where M is even.

Now suppose that $M = 2N + 1$ for some $N \in \mathbb{N}$, and that $\ell^\infty(P_{2N+1})$ is C-projective in $\ell^1(P_{2N+1})$-**mod**. There is an isometric identification of Banach algebras $\ell^1(P_{2N+1}) = \ell^1(P_{2N}) \widehat{\otimes} \ell^1(P_1)$. By Proposition 2.2.12 $\ell^\infty(P_{2N})$ is C-projective in $\ell^1(P_{2N})$-**mod**. Therefore $C \geq 2^N = 2^{\lfloor M/2 \rfloor}$. □

Remark 7.4.13. We have in fact proved a slightly stronger result than Theorem 7.4.12. We have shown that there exists $y \in \ell^\infty(P_M)$, such that

$$\inf_\rho \|\rho(y)\| \geq 2^{\lfloor M/2 \rfloor} ,$$

where the infimum is taken over all left $\ell^1(P_M)$-module morphisms $\rho : \ell^\infty(P_M) \to \ell^1(P_M) \widehat{\otimes} \ell^\infty(P_M)$ with $\pi \circ \rho = I_{\ell^\infty(P_M)}$.

Theorem 7.4.14. *Let X be an infinite set. Then the module $\ell^1(P_X)$ is not injective in* **mod**-$\ell^1(P_X)$*, and the module $\ell^1(F_X)$ is not injective in* **mod**-$\ell^1(F_X)$.

Proof. Assume towards a contradiction that $\ell^1(P_X)$ is C-right injective for some $C > 1$. Take $N \in \mathbb{N}$ and a subset $X_N \subset X$ with $|X_N| = 2N$. There is an isometric identification of Banach algebras $\ell^1(P_X) = \ell^1(P_N) \widehat{\otimes} \ell^1(P_{X\setminus X_N})$. By Proposition 2.2.12 $\ell^1(P_{2N})$ is C-right injective in **mod**-$\ell^1(P_{2N})$. By Theorem 7.4.12, $2^N \leq C$. This is true for each $N \in \mathbb{N}$, the required contradiction.

The same argument, but using the identification $\ell^1(F_X) = \ell^1(P_N) \widehat{\otimes} \ell^1(F_{X\setminus X_N})$, shows that $\ell^1(F_X)$ is not injective in **mod**-$\ell^1(F_X)$. □

We find the following estimate about alternating series interesting and difficult to prove directly.

Corollary 7.4.15. *Let $M \in \mathbb{N}$. Then*

$$2^{\lfloor 3M/2 \rfloor} \leq \sup_\gamma \sum_{s\in P_M} \left| \sum_{t\in P_M} \gamma_t (-1)^{t\cap s} \right| \leq 2^{3M/2} ,$$

where the supremum is taken over all complex sequences $\gamma = \{\gamma_t : t \in P_M\} \subset \overline{\mathbb{D}}$.

Proof. Let ρ be the map defined in Theorem 7.4.6. The proof of Theorem 7.4.6 shows that for $y = \sum_{t \in P_M} (-1)^{|t|} \gamma_t \lambda_t \in \ell^\infty(P_M)$ we have

$$\|\rho(y)\| = \frac{1}{2^M} \sum_{s \in P_M} \left| \sum_{t \in P_M} \gamma_t (-1)^{t \cap s} \right| .$$

We have also shown that $2^{\lfloor M/2 \rfloor} \leq \|\rho\| \leq 2^{M/2}$. The result follows. □

7.5 Open questions

Question 7.5.1. *For which semigroups S is $\ell^1(S)$ a right injective $\ell^1(S)$-module?*

Question 7.5.2. *For which weakly right cancellative semigroups S is $c_0(S)$ a projective left $\ell^1(S)$-module? Under the assumption that S is also weakly right cancellative, then projectivity of $c_0(S)$ implies that S is finite. Can we remove this assumption? For which finite semigroups is the module $c_0(S)$ projective?*

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